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RESEARCH REPORT No. EM-87

A New Method for Calculating Scattering, with Particular Reference to the Circular Disc

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CONTRACT NO. AF 19(122)-42
DECEMBER 1955

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A NEW METHOD FOR CALCULATING SCATTERING, WITH PARTICULAR REFERENCE TO THE CIRCULAR DISC

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The research reported in this document has been made possible through support and sponsorship extended by the Air Force Cambridge Research Center, under Contract No. AF 19(122)-42. It is published for technical information only, and does not necessarily represent recommendations or conclusions of the sponsoring agency.

New York, 1955

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Abstract

It is shown that a solution in closed form can be obtained for the problem of finding the pressure distribution on a circular disc in an arbitrary axi-symmetrical sound wave when it is assumed that the scattered field behaves like a standing wave at infinity and approaches the static limit as the wavelength becomes infinitely long. This result is used to convert the standard integral equation of the first kind, which arises when the scattered field behaves like a radiating wave, into an integral equation of the second kind.

The new integral equation has a particularly simple kernel so that, although it is difficult to solve exactly, it can be solved approximately with ease. It is found that one approximation leads to an expression (in terms of well-tabulated functions) for the scattering coefficient which agrees adequately with the values obtained from the spheroidal function expansion provided that the product of the wave number and radius of the disc is less than 3. The approximation is capable of systematic extension to cover larger values of the product of wave number and radius. The approximation is also used to determine the scattering coefficient when the incident field is derived from a point source on the axis of the disc.

A general theory is developed of the corresponding integral equation which would arise in any scattering problem. It is shown that upper bounds can be obtained for the scattered amplitude and the scattering coefficient. Numerical values are given for a circular disc in a normally incident plane sound wave and in the field due to an axial point source.

Further, it is shown that it is always possible to estimate the error made in finding the source distribution by solving the integral equation approximately. From this, estimates of the error in the scattered amplitude and scattering coefficient are deduced. Numerical results are given for a circular disc in a normally incident plane sound wave.

Introduction

The problem of the diffraction by a circular disc or aperture has attracted much attention in recent years. It is the simplest problem in which there is a diffracting edge of finite length and therefore serves as a convenient test for trying out methods which may be applicable to more complicated problems. These methods have been reviewed by Bouwkamp[1], so that here it is necessary only to describe briefly one or two of them. The direct method of separation of variables leads a solution in spheroidal functions which was given by Bouwkamp[2]. In principle, all that is required can be obtained provided that sufficient computation and information about spheroidal functions is available. To date, numerical results have been given in the range $0 \leq \alpha \leq 10$, where α is the product of the wave number and the radius of the disc. Thus, in practice, the method of separation of variables is most useful at the low-frequency end of the frequency scale. It is, however, applicable only to special shapes. In an attempt to provide a method which was not restricted to shape, Levine and Schwinger[3] introduced a variational principle. They were able to obtain good agreement with Bouwkamp, but it has been shown (Jones[4]) that, in general, only a special choice of trial function (which coincides with Levine and Schwinger's choice in the problem of the circle) can lead to accuracy (in a certain sense) at low frequencies. The determination of the special trial function is equivalent to solving the problem by means of a power series expansion in terms of the parameter α. The expansion correct to terms of $O(\alpha^8)$ was derived by Bouwkamp[1]. The range of validity of such an expansion is not known, although there is some indication from the following work and that of Magnus[5] that it does not extend much beyond $\alpha = 1$. Another method, using an assumed aperture distribution in terms of hypergeometric polynomials, has been discussed by Chako[6] and by Nomura and Katsura[7]. The application of the Wiener-Hopf technique has been considered by Leitner and Wells[8] and leads to mathematical difficulties similar to those encountered in diffraction by a finite waveguide Jones[9].

At high frequencies there is no exact mathematical solution which gives results correct to 0.1% (say) in practice, although in principle these could be obtained from the spheroidal function solution. There are available, however, approximations based on physical reasoning. The classical method is due to Kirchhoff, but more convincing explanations of what happens have been given by Clemmow*[10] and Keller[11]. Their explanations have the virtues of being easy to understand, of leading to relatively simple analysis and computation, and of giving a scattering cross section which agrees with that given by the spheroidal function solution at intermediate frequencies.

It is desirable to have a similarly simple approach at low frequencies since all the methods described require complicated mathematics and considerable computation. A classical method, due to Rayleigh, based on a static approximation requires only simple analysis, but, unfortunately, has only a very restricted range of applicability. The work that follows is an attempt to simplify as many aspects as possible in the low-frequency problem. A comparatively simple physical picture is employed; the ensuing analysis is complicated but enables one to reach a stage where the computation is simple. The mathematical technique has been developed only for the circular disc but it is hoped that it can be generalized to more general obstacles.

The physical picture employed is a generalization of Rayleigh's static picture. In describing it, it is convenient to use the terminology of sound waves. At low frequencies the field is essentially static in any domain whose linear dimensions are much smaller than a wavelength and there is very little radiation from the disc. If, therefore, we solve the problem when there is no radiation we should obtain a very good approximation to the pressure distribution on the disc. We can then construct a radiation field which has the same pressure distribution; this should be an excellent low-frequency approximation to the field. Clearly we could devise an iteration scheme based on these ideas which would improve the accuracy and extend the frequency range of applicability. The difficulties involved in the discussion of the convergence of an iteration scheme are well known and it is therefore desirable that the mathematical formulation shall be such that these difficulties either do not occur or can be overcome.

---

* Clemmow discusses electromagnetic phenomena, but the fundamental principles are applicable to scalar problems.

It must be realized that the problem in which there is no radiation is not the static one; it is the problem in which the waves at infinity are standing waves. The approximation of standing waves is much less drastic than the static approximation because the field still satisfies the harmonic wave equation although it is non-radiating.

Accordingly, in the mathematical formulation, we split the scattered field into a non-radiating field which becomes the static field in the limit of zero frequency, and the remainder of the field. This remainder is then added to the incident field and the sum is treated as the incident field in the non-radiating problem. This problem is then solved exactly by means of integral equations and leads to a Fredholm integral equation governing the original problem. In this way the question of the convergence of the earlier iteration scheme is avoided. The iteration is obviously equivalent to solving the Fredholm equation by a Neumann expansion, so that, if we wish to use it, we can quote well-known theorems to say when it will converge.

The mathematical difficulty mentioned earlier arises in the exact solution of the non-radiating problem and the method can be generalized to other shapes only if this solution can be obtained in general. It should be remarked that the method can be used for the diffraction of electromagnetic waves by a circular disc; this will be discussed in a future paper.

The Fredholm integral equation has many advantages. The power series solution can be obtained from it very rapidly. It is easy to solve approximately; one obtains a pressure distribution which can be computed easily from well tabulated functions. Further, it is possible to estimate the error made in solving approximately a Fredholm integral equation, so that one can state, for example, that our value for the scattering coefficient is correct to so many percent. Also it is possible to deduce an upper bound for the scattering coefficient. A Fredholm integral equation has been derived by Magnus[5], but it is much more complicated than the one we obtain.

The following analysis is restricted to the case when the pressure distribution on the disc depends only on the radius. The resulting integral equation is formulated in Section 1. In Section 2 the angular integration is performed, and in Section 3 the Fredholm integral is derived. The problem of the incident plane wave is discussed in Section 4 and an approximate solution, agreeing well with the spheroidal function solution, is obtained in Section 5. Section 6 is devoted to the problem when the incident wave is supplied by a point source.

In Section 7 we give a general theory, applicable to any obstacle, which shows how an upper bound for the scattering coefficient may be obtained. Section 8 contains an estimate of the error introduced by our method of approximation.

The derivation of a number of results has been carried out in appendices. References to equations in appendices are written for example, as (B1), meaning equation (1) of Appendix B.

1. The integral equation

We consider the problem of the diffraction of a harmonic sound wave by a rigid circular disc of radius a. The boundary condition is that the normal derivative of the field must vanish on the disc. The case when the field vanishes on the disc could also be treated by the method we shall give but will not be considered here. Results for the circular aperture in a plane screen may be deduced from those for the circular disc by Babinet's principle.

Let ρ, ϕ and z be cylindrical polar coordinates, and normalize all lengths by division by a so that the disc occupies $0 \leq \rho \leq 1$, $z = 0$. Let the incident field be independent of ϕ and be represented by $u_o(\rho,z)$ at (ρ,ϕ,z), the time dependence $e^{i\omega t}$ being understood. The total field $u(\underline{r})$, at the point whose position vector with respect to the origin is $\underline{r}$, is then independent of ϕ. Such a field occurs for a normally incident plane wave and when the incident wave is due to a source on $\rho = 0$. The problem of the unsymmetrical incident field will be considered in a separate work.

The total field at the point $\underline{r}_1$ can be represented by

$$u(\underline{r}_1) = u_o(\rho_1,z_1) + \frac{1}{4\pi} \int_S g(\underline{\rho}) \left[\frac{\partial}{\partial z} \Psi(\underline{r},\underline{r}_1)\right]_{z=0} dS$$

where S is the area of the unit circle, $\underline{\rho}$ is a point of S, $\Psi = e^{-i\alpha|r_1-\underline{r}|}/|\underline{r}_1-\underline{r}|$, and α is the (dimensionless) product of the wave number and a. To ensure a unique solution (Jones[12]) we have to impose the edge condition that $g = 0$ on the boundary of S.

The boundary condition that has to be satisfied on S is $\partial u/\partial z_1 = 0$. To avoid non-integrable singularities in the integral we modify the expression for $\partial u/\partial z_1$. We have

$$\frac{\partial u}{\partial z_1} = \frac{\partial}{\partial z_1} u_o(\rho_1,z_1) + \frac{1}{4\pi}\int_S g(\underline{\rho}) \left[\frac{\partial^2}{\partial z_1 \partial z} \Psi(\underline{r},\underline{r}_1)\right]_{z=0} dS$$

$$= \frac{\partial}{\partial z_1} u_o(\rho_1,z_1) + \frac{1}{4\pi}\int_S g(\underline{\rho}) \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \alpha^2\right) \Psi(\underline{\rho},\underline{r}_1)\, dS$$

since $\partial\Psi/\partial z = -\partial\Psi/\partial z_1$ and $(\nabla^2 + \alpha^2)\Psi = 0$, where ∇^2 is the three-dimensional Laplacian. Since $g = 0$ on the boundary of S the divergence theorem gives

$$\frac{\partial u}{\partial z_1} = \frac{\partial}{\partial z_1} u_o(\rho_1, z_1) + \frac{1}{4\pi} \int_S \left\{ \alpha^2 g(\underline{\rho}) \Psi(\underline{\rho}, \underline{r}_1) - \text{grad } g \cdot \text{grad } \Psi(\underline{\rho}, \underline{r}_1) \right\} dS$$

$$= \frac{\partial}{\partial z_1} u_o(\rho_1, z_1) + \frac{1}{4\pi} \text{div}_1 \int_S \Psi(\underline{\rho}, \underline{r}_1) \text{grad } g dS + \frac{\alpha^2}{4\pi} \int_S g(\underline{\rho}) \Psi(\underline{\rho}, \underline{r}_1) dS$$

where the operators grad and div are independent of z, the suffix 1 indicating operations with respect to $\underline{\rho}_1$. There is now no difficulty in taking the limit as $z_1 \to 0$ and we obtain the following equation to determine g

$$(1) \qquad - \frac{\partial}{\partial z_1} u_o(\rho_1, 0) = \frac{1}{4\pi} \text{div}_1 \int_S \Psi(\underline{\rho}, \underline{\rho}_1) \text{ grad } g \, dS + \frac{\alpha^2}{4\pi} \int_S g(\underline{\rho}) \Psi(\underline{\rho}, \underline{\rho}_1) dS,$$

where $\underline{\rho}_1$ is a point of S.

Since g is independent of ϕ, only Ψ is involved in the integration with respect to ϕ. We proceed to the consideration of this in the next section.

2. The integration with respect to ϕ

In the first place we shall consider only the real part of Ψ. Then, it is necessary to evaluate integrals of the type

$$I = \int_0^{2\pi} \cos(\phi - \beta) \frac{\cos \gamma \left\{ 1 + b^2 - 2b \cos(\phi - \phi_1) \right\}^{1/2}}{\left\{ 1 + b^2 - 2b \cos(\phi - \phi_1) \right\}^{1/2}} d\phi,$$

where β, γ and b (< 1) are arbitrary real constants, since

$$|\underline{\rho} - \underline{\rho}_1| = \left\{ \rho^2 + \rho_1^2 - 2\rho\rho_1 \cos(\phi - \phi_1) \right\}^{1/2} .$$

Now

$$I = 4\cos(\phi_1-\beta)\int_0^{\pi/2} \cos 2\phi \; \frac{\cos\gamma(1+b^2-2b\cos 2\phi)^{1/2}}{(1+b^2-2b\cos 2\phi)^{1/2}}\,d\phi$$

$$= 2\cos(\phi_1-\beta)\int_0^{\pi} \frac{\left\{\cos\theta(1-b^2\sin^2\theta)^{1/2}+b\sin^2\theta\right\}\cos\gamma\left|(1-b^2\sin^2\theta)^{1/2}-b\cos\theta\right|}{(1-b^2\sin^2\theta)^{1/2}}\,d\theta$$

on putting $\tan\theta = \frac{\sin 2\phi}{\cos 2\phi - b}$. Hence

$$(2)\quad I = \frac{4}{b}\cos(\phi_1-b)\int_0^b \left\{ \frac{x^2\cos\gamma(1-x^2)^{\frac{1}{2}}\cos\gamma(b^2-x^2)^{\frac{1}{2}}}{(1-x^2)^{1/2}\,(b^2-x^2)^{1/2}} + \sin\gamma(1-x^2)^{\frac{1}{2}}\sin\gamma(b^2-x^2)^{\frac{1}{2}} \right\} dx.$$

Similarly

$$(3)\quad \int_0^{2\pi} \frac{\cos\gamma\left\{1+b^2-2b\cos(\phi-\phi_1)\right\}^{1/2}}{\left\{1+b^2-2b\cos(\phi-\phi_1)\right\}^{1/2}}\,d\phi = 4\int_0^b \frac{\cos\gamma(1-x^2)^{1/2}\cos\gamma(b^2-x^2)^{1/2}}{(1-x^2)^{1/2}\,(b^2-x^2)^{1/2}}\,dx\,.$$

Similar results may be obtained for the integration of the imaginary part of Ψ with respect to ϕ but, for the form in which we want them, it is quicker to proceed in a different fashion. We have

$$\int_0^{2\pi} \frac{\sin\alpha|\underline{\rho}-\underline{\rho}_1|}{|\underline{\rho}-\underline{\rho}_1|}\,d\phi = \left(\frac{\alpha\pi}{2}\right)^{1/2}\int_0^{2\pi} \frac{J_{1/2}(\alpha|\underline{\rho}-\underline{\rho}_1|)}{|\underline{\rho}-\underline{\rho}_1|^{1/2}}\,d\phi,$$

where J_ν is the Bessel function of the first kind and ν-th order. On using Sonine's first finite integral (Watson[13]) the integral on the right may be written

$$\alpha\int_0^{2\pi}\int_0^{\pi/2} \sin\theta\, J_0(\alpha|\underline{\rho}-\underline{\rho}_1|\sin\theta)\,d\theta\,d\phi$$

$$= 2\pi\alpha\int_0^{\pi/2} \sin\theta\, J_0(\alpha\rho\sin\theta)\,J_0(\alpha\rho_1\sin\theta)\,d\theta$$

after a use of the addition theorem for Bessel functions. Hence

$$\operatorname{div}_1 \int_S \frac{\sin\alpha|\underline{\rho}-\underline{\rho}_1|}{|\underline{\rho}-\underline{\rho}_1|} \operatorname{grad} g \, dS + \alpha^2 \int_S \frac{g \sin\alpha|\underline{\rho}-\underline{\rho}_1|}{|\underline{\rho}-\underline{\rho}_1|} dS$$

$$= \int_0^1 \rho g'(\rho)\, 2\pi\alpha^2 \int_0^{\pi/2} \sin^2\theta \, J_1(\alpha\rho\sin\theta)\, J_o(\alpha\rho_1\sin\theta) d\theta d\rho$$

$$+ 2\pi\alpha^3 \int_0^1 \rho g(\rho) \int_0^{\pi/2} \sin\theta \, J_o(\alpha\rho\sin\theta)\, J_o(\alpha\rho_1\sin\theta) d\theta d\rho$$

$$= 2\pi\alpha^3 \int_0^1 \rho g(\rho) \int_0^{\pi/2} \sin\theta \cos^2\theta \, J_o(\alpha\rho\sin\theta) J_o(\alpha\rho_1\sin\theta) d\theta d\rho$$

$$(4) \qquad = \frac{2\pi\alpha^2}{\rho_1} \frac{d}{d\rho_1} \int_0^1 \rho_1 \rho g(\rho) \int_0^{\pi/2} \cos^2\theta \, J_o(\alpha\rho\sin\theta) J_1(\alpha\rho_1\sin\theta) d\theta d\rho \; .$$

Now, on putting $\lambda = 1$, $\mu = \frac{1}{2}$ and $\nu = \frac{1}{2}$ in the formula (C.1) given in Appendix C, we have

$$(5) \quad \int_0^{\pi/2} J_o(z\sin\theta) J_1(z\sin\theta)\cos^2\theta d\theta = \frac{1}{Z} \int_0^Z x^2 \frac{J_{-1/2}\left\{(Z^2-x^2)^{1/2}\right\}}{(Z^2-x^2)^{1/4}} \frac{J_{3/2}\left\{(z^2-x^2)^{1/2}\right\}}{(z^2-x^2)^{3/4}} dx.$$

Since

$$\frac{x\, J_{3/2}\left\{(z^2-x^2)^{1/2}\right\}}{(z^2-x^2)^{3/4}} = \frac{d}{dx} \frac{J_{1/2}\left\{(z^2-x^2)^{1/2}\right\}}{(z^2-x^2)^{1/4}} \; ,$$

the use of (2), (3), (4), and (5) in (1) leads, after some straightforward manipulation, to the equation

$$-\frac{\partial}{\partial z_1} u_o(\rho_1,0) = \frac{1}{\pi\rho_1} \frac{d}{d\rho_1} \int_0^1 g'(\rho) \int_0^{\min(\rho,\rho_1)} \frac{t^2 \cos\left\{\alpha(\rho_1^2-t^2)^{\frac{1}{2}}\right\} \cos\left\{\alpha(\rho^2-t^2)^{\frac{1}{2}}\right\}}{(\rho_1^2-t^2)^{1/2}\,(\rho^2-t^2)^{1/2}} dt d\rho$$

$$-\frac{i}{\pi\rho_1} \frac{d}{d\rho_1} \int_0^1 \rho g(\rho) \int_0^{\rho_1} \frac{t\cos\left\{\alpha(\rho_1^2-t^2)^{\frac{1}{2}}\right\}}{(\rho_1^2-t^2)^{1/2}} \frac{d}{dt} \frac{\sin\left\{\alpha(\rho^2-t^2)^{\frac{1}{2}}\right\}}{(\rho^2-t^2)^{1/2}} dt$$

or

$$\begin{aligned}-\frac{\partial}{\partial z_1} u_o(\rho_1,0) = \frac{1}{\pi\rho_1}\,\frac{d}{d\rho_1}\int_0^{\rho_1} \frac{\cos\left\{\alpha(\rho_1^2-t^2)^{1/2}\right\}}{(\rho_1^2-t^2)^{1/2}}\Bigg[\int_t^1 \frac{t^2 g'(\rho)\cos\left\{\alpha(\rho^2-t^2)^{1/2}\right\}}{(\rho^2-t^2)^{1/2}}\,d\rho \\ -i\int_0^1 \rho t g(\rho)\,\frac{d}{dt}\,\frac{\sin\left\{\alpha(\rho^2-t^2)^{1/2}\right\}}{(\rho^2-t^2)^{1/2}}\,d\rho\Bigg]\,dt\ .\end{aligned} \tag{6}$$

The solution of (6) is considered in the next section.

3. The solution of the transformed equation

The equation (6) is of the form

$$f(\rho_1) = \frac{1}{\pi\rho_1}\,\frac{d}{d\rho_1}\int_0^{\rho_1} h(t)\,\frac{\cos\left\{\alpha(\rho_1^2-t^2)^{1/2}\right\}}{(\rho_1^2-t^2)^{1/2}}\,dt\ .$$

The solution of an equation of this type is given in Appendix A (see (A.4)). Identifying h(t) with its appropriate counterpart in (6) we have

$$\begin{aligned}x\int_x^1 g'(\rho)\,\frac{\cos\left\{\alpha(\rho^2-x^2)^{1/2}\right\}}{(\rho^2-x^2)^{1/2}}\,d\rho - i\int_0^1 \rho g(\rho)\,\frac{d}{dx}\,\frac{\sin\left\{\alpha(\rho^2-x^2)^{1/2}\right\}}{(\rho^2-x^2)^{1/2}}\,d\rho \\ = -2\int_0^x \frac{\partial}{\partial z_1}u_o(t)\,\frac{t\cosh\left\{\alpha(x^2-t^2)^{1/2}\right\}}{(x^2-t^2)^{1/2}}\,dt\end{aligned}$$

or

$$\begin{aligned}\int_x^1 g'(\rho)\,\frac{\cos\left\{\alpha(\rho^2-x^2)^{1/2}\right\}}{(\rho^2-x^2)^{1/2}}\,d\rho + i\int_0^1 g(\rho)\,\frac{d}{d\rho}\,\frac{\sin\left\{\alpha(\rho^2-x^2)^{1/2}\right\}}{(\rho^2-x^2)^{1/2}}\,d\rho \\ = -\frac{2}{x}\int_0^x \frac{\partial}{\partial z_1}u_o(t)\,\frac{t\cosh\left\{\alpha(x^2-t^2)^{1/2}\right\}}{(x^2-t^2)^{1/2}}\,dt\ .\end{aligned} \tag{7}$$

There is a number of ways of rewriting (7). We will consider the method which is applicable to low frequencies or small values of α. The first term of the left-hand side was derived from the part of the original integral equation which corresponds to standing waves at infinity and which goes over to the static limit as $\alpha \rightarrow 0$. Following the principle described in the Introduction we regard (7) as an equation in which the first term on the left-hand side is equal to a known function. The solution of such an equation with the condition $g(1) = 0$ is given in Appendix A (A.5) and applying this solution to (7) we obtain

$$
\begin{aligned}
g(x) = -\frac{2}{\pi}\int_x^1 \frac{t\cosh\left\{\alpha(t^2-x^2)^{1/2}\right\}}{(t^2-x^2)^{1/2}} \Bigg[& -\frac{2}{t}\int_0^t \frac{\partial}{\partial z_1} u_o(\rho)\, \frac{\rho\cosh\left\{\alpha(t^2-\rho^2)^{1/2}\right\}}{(t^2-\rho^2)^{1/2}}\, d\rho \\
& - i\int_0^1 g(\rho)\, \frac{d}{d\rho}\, \frac{\sin\left\{\alpha(\rho^2-t^2)^{1/2}\right\}}{(\rho^2-t^2)^{1/2}}\, d\rho \Bigg]\, dt\, . \qquad (8)
\end{aligned}
$$

This equation holds for $0 \leq x \leq 1$. Now

$$
\int_x^1 \frac{t\cosh\left\{\alpha(t^2-x^2)^{1/2}\right\}}{(t^2-x^2)^{1/2}} \int_0^1 g(\rho)\, \frac{d}{d\rho}\, \frac{\sin\left\{\alpha(\rho^2-t^2)^{1/2}\right\}}{(\rho^2-t^2)^{1/2}}\, d\rho\, dt
$$

$$
= -\int_0^1 g\left\{(1-\rho^2)^{1/2}\right\} \frac{d}{d\rho} \int_0^{(1-x^2)^{1/2}} \frac{t\sin\left\{\alpha(t^2-\rho^2)^{1/2}\right\}}{(t^2-\rho^2)^{1/2}}\, \frac{\cosh\left\{\alpha(1-x^2-t^2)^{1/2}\right\}}{(1-x^2-t^2)^{1/2}}\, dt\, d\rho,
$$

on inverting the order of integration and making obvious changes in the integrating variables. Also

$$
\int_0^y \frac{t\sin\left\{\alpha(t^2-\rho^2)^{1/2}\right\}}{(t^2-\rho^2)^{1/2}}\, \frac{\cosh\left\{\alpha(y^2-t^2)^{1/2}\right\}}{(y^2-t^2)^{1/2}}\, dt
$$

$$
= \frac{1}{2}\int_0^y \frac{\sinh\left\{\alpha(\rho^2-t^2)^{1/2} + \alpha(y^2-t^2)^{1/2}\right\} + \sinh\left\{\alpha(\rho^2-t^2)^{1/2} - \alpha(y^2-t^2)^{1/2}\right\}}{(\rho^2-t^2)^{1/2}\,(y^2-t^2)^{1/2}}\, t\,dt
$$

$$(9) \qquad = -\frac{1}{2}\left\{ \int_{\rho+y}^{(\rho^2-y^2)^{1/2}} \frac{\sinh \alpha t}{t}\, dt - \int_{\rho-y}^{(\rho^2-y^2)^{1/2}} \frac{\sinh \alpha t}{t}\, dt \right\}.$$

Hence (8) becomes

$$(10) \qquad G(y) = F(y) + \frac{1}{\pi}\int_0^1 G(x)\left\{ \frac{\sinh \alpha(x-y)}{x-y} - \frac{\sinh \alpha(x+y)}{x+y} \right\} dx \quad (0 \leq y \leq 1),$$

where

$$(11) \qquad G(y) = g\left\{ (1-y^2)^{1/2} \right\}$$

and

$$(12) \qquad F(y) = \frac{4}{\pi}\int_{(1-y^2)^{1/2}}^{1} \frac{\cosh\left\{\alpha(t^2-1+y^2)^{1/2}\right\}}{(t^2-1+y^2)^{1/2}} \int_0^t \frac{\partial}{\partial z_1} u_0(\rho)\rho\, \frac{\cosh\left\{\alpha(t^2-\rho^2)^{1/2}\right\}}{(t^2-\rho^2)^{1/2}}\, dt d\rho \, .$$

It should be remarked that the method of deriving (10) shows that the unique solution of (1) is a solution of (10). Since, however, the eigenvalues of (10) are real, the theory of the Fredholm integral equation implies that (10) possesses a unique continuous solution. The solution of (10) can, therefore, be identified with that of (1).

We shall now consider equation (10) for two particular incident fields.

4. The incident plane wave

Suppose now that the incident field is a plane wave so that

$$u_0(\rho,z) = e^{-i\alpha z} .$$

Then

$$\frac{\partial}{\partial z_1} u_0(\rho,0) = -i\alpha$$

and we obtain

$$F(y) = -\frac{4i}{\pi}\int_{(1-y^2)^{1/2}}^{1} \frac{\cosh\left\{\alpha(t^2-1+y^2)^{1/2}\right\}}{(t^2-1+y^2)^{1/2}} \sinh \alpha t \ dt$$

$$= -\frac{2i}{\pi}\int_{1-y}^{1+y} \frac{\sinh \alpha x}{x}\, dx$$

in the same way as was used in deriving (9). Hence (10) becomes, for an incident plane wave,

$$(13) \quad G(y) = -\frac{2i}{\pi}\int_{1-y}^{1+y} \frac{\sinh \alpha x}{x}\, dx + \frac{i}{\pi}\int_{0}^{1} G(x)\left\{\frac{\sinh \alpha(x-y)}{x-y} - \frac{\sinh \alpha(x+y)}{x+y}\right\} dx \quad (0 \leq y \leq 1).$$

Observe that the right-hand side of (13) vanishes when $y = 0$, as it should. In fact the right-hand side is an odd function of y. Therefore we can define $G(-y)$ when y is positive to be $-G(y)$; equation (13) then becomes

$$(14) \qquad G(y) = -\frac{2i}{\pi}\int_{1-y}^{1+y} \frac{\sinh \alpha x}{x}\, dx + \frac{i}{\pi}\int_{-1}^{1} G(x)\frac{\sinh \alpha(x-y)}{x-y}\, dx \quad (-1 \leq y \leq 1).$$

This Fredholm integral equation has to be solved to complete the solution of our problem. There are several methods available and the application of some of them will now be discussed.

Firstly, an exact solution of (14) may be written down by means of the Fredholm theory, but the form of the kernel makes the calculation of the resolvent impractical.

Secondly, we could find a solution by expanding in terms of the eigenfunctions of the kernel. The eigenfunctions are, however, spheroidal functions (Meixner and Schäfke [14]), and so this method appears to have no advantage over separating the harmonic wave equation in spheroidal coordinates.

Thirdly, we could attempt to form a solution by means of iteration. The simplest way of doing this is to take $-\frac{2i}{\pi}\int_{1-y}^{1+y} \frac{\sinh \alpha x}{x}\, dx$ as the first approximation and substitute successively in the integral. This is an excellent method when α is small because the kernel starts at the term involving α^3 since G(x) is odd. Hence, since the first approximation starts at α, the first approximation to G(y) is correct to $O(\alpha^3)$, the second to $O(\alpha^6)$ and so on. It is easily verified that the expansion in powers of α is the same as that obtained by Bouwkamp by substituting a power series expansion for g in (1). However, the iterated solution of (14) provides the coefficients in the expansion much more rapidly than his method. Two iterations can be carried out very quickly. Since the resulting pressure distribution contains more terms than given by Bouwkamp it is worthwhile to reproduce it. It is

$$
\begin{aligned}
G(y) = -\frac{4i\alpha y}{\pi}\Bigg\{ & 1 + \frac{\alpha^2}{6} - \frac{2i\alpha^3}{9\pi} + \frac{\alpha^4}{120} - \frac{13i\alpha^5}{225\pi} + \left(\frac{1}{5040} - \frac{4}{81\pi^2}\right)\alpha^6 - \frac{323i\alpha^7}{44100\pi} \\
& + \left(\frac{1}{362880} - \frac{38}{2025\pi^2}\right)\alpha^8 + \dots \Bigg\} \\
& - \frac{2i\alpha^3 y^3}{9\pi}\left\{1 + \frac{3\alpha^2}{10} - \frac{2i\alpha^3}{5\pi} + \frac{\alpha^4}{56} - \frac{19i\alpha^5}{175\pi} + \left(\frac{1}{4260} - \frac{4}{45\pi^2}\right)\alpha^6 + \dots\right\} \\
& - \frac{2i\alpha^5 y^5}{300\pi}\left\{1 + \frac{5\alpha^2}{14} - \frac{10i\alpha^3}{21\pi} + \frac{5\alpha^4}{216} + \dots\right\} \\
& - \frac{2i\alpha^7 y^7}{17640\pi}\left\{1 + \frac{7\alpha^2}{18} + \dots\right\} - \frac{2i\alpha^9 y^9}{1632960\pi}\left\{1 + \dots\right\} .
\end{aligned}
$$

For comparison with Bouwkamp's formula it must be remembered that his distribution is for the complementary aperture problem and is therefore $-\frac{1}{2}$ times ours, also he uses the time dependence $e^{-i\omega t}$.

It is known that the iterated solution described fails to converge when $1/\pi$ exceeds the first eigenvalue, i.e., it fails to converge when α exceeds some constant. This probably means that the power series expansion also diverges when α exceeds this value, but it may mean only that the expansion has been rearranged in a divergent series of groups of terms. The iterated solution converges if

$$(15) \qquad \frac{1}{\pi^2}\int_{-1}^{1}\int_{-1}^{1} \frac{\sinh^2 \alpha(x-y)}{(x-y)^2}\, dx\, dy < 1,$$

which is certainly true if $\sinh 2\alpha < \pi$. It does not, of course, follow that the power series expansion converges for this range of α.

More complicated methods of iteration are available which converge everywhere (Brückner[15]) but no attempt has been made to use then from equation (14). Instead we have solved (14) approximately in a straightforward manner which leads to tabulated functions, enables G to be computed for moderate values of α fairly rapidly, and provides an estimate of the error.

5. Solution by approximation to the kernel

Since $(\sinh \alpha z)/z$ is an integral function it can be expanded in a Taylor series about any point. We deduce at once that

$$G(y) = -\frac{2i}{\pi}\int_{1-y}^{1+y} \frac{\sinh \alpha x}{x}\, dx + \sum_{n=0}^{\infty} \frac{a_n \alpha^{2n} y^{2n+1}}{(2n+1)!} .$$

The substitution of this representation in (14) leads to the equations

$$(16) \qquad + \frac{\pi i a_m \alpha^{2m}}{(2m+1)!} = -2iK_m/\pi + \sum_{n=0}^{\infty} A_{mn}\, a_n \qquad (m = 0, 1, \ldots)$$

to determine the constants a_m. Here

$$A_{mn} = \frac{\alpha^{2n}}{(2m+1)!(2n+1)!}\int_{-1}^{1} x^{2n+1} \frac{d^{2m+1}}{dx^{2m+1}} \frac{\sinh \alpha x}{x}\, dx$$

and

$$K_m = \frac{1}{(2m+1)!}\int_{-1}^{1}\int_{1-x}^{1+x} \frac{\sinh \alpha t}{t} \frac{d^{2m+1}}{dx^{2m+1}} \frac{\sinh \alpha x}{x}\, dt\, dx.$$

It is shown in Appendix B that

$$A_{mn} = A_{nm} = \frac{1}{(2m+1)!(2n+1)!(n-m)} \left[\left\{ (2n+1)! \sum_{s=0}^{n} \frac{\alpha^{2m+2s}}{(2s)!} - (2m+1)! \sum_{s=0}^{m} \frac{\alpha^{2n+2s}}{(2s)!} \right\} \sinh \alpha \right.$$

$$\left. - \left\{ (2n+1)! \sum_{s=0}^{n-1} \frac{\alpha^{2m+2s+1}}{(2s+1)!} - (2m+1)! \sum_{s=0}^{m-1} \frac{\alpha^{2n+2s+1}}{(2s+1)!} \right\} \cosh \alpha \right], \qquad (m \neq n)$$

and

$$A_{mm} = \frac{2\alpha^{2m}}{(2m+1)!} \left[\sinh \alpha \sum_{s=0}^{m} \frac{\alpha^{2s}}{(2s)!} \left\{ \psi(2m+2) - \psi(2s+1) \right\} - \frac{1}{2} \left\{ \mathrm{Ei}(\alpha) - \mathrm{Ei}(-\alpha) \right\} \right.$$

$$\left. - \cosh \alpha \sum_{s=0}^{m-1} \frac{\alpha^{2s+1}}{(2s+1)!} \left\{ \psi(2m+2) - \psi(2s+2) \right\} \right]$$

where $\psi(m) = 1 + \frac{1}{2} + \dots + \frac{1}{m-1} - \gamma$, $\psi(1) = -\gamma$, $\gamma = 0.5772 \dots$ and

$$\mathrm{Ei}(x) = \int_{-\infty}^{x} \frac{e^{t}}{t} \, dt.$$

The scattering cross section of an obstacle with an incident plane wave is $-(4\pi/k)\mathcal{I}A$, where A is the amplitude of the scattered wave in the direction of propagation of the incident wave and k is the wave number. Thus the scattering cross section of the disc is

$$-2\pi a^2 \mathcal{R} \int_{0}^{1} g\rho \, d\rho \, .$$

Hence the scattering coefficient (the cross section divided by the area of the disc) is

$$-2 \mathcal{R} \sum_{n=0}^{\infty} \frac{a_n \alpha^{2n}}{(2n+1)!(2n+3)} \, .$$

The complete solution of (16) is scarcely practicable but an approximate solution, in which the infinite series in G is truncated to form a finite series, can be provided with reasonable computation when the truncated series does not involve too many terms. A method for calculating the error produced by this truncation is given in Section 8.

There is no possibility that the determinant of the coefficients vanishes either for the complete or truncated series. If the determinant did vanish, i/π would be an eigenvalue of the kernel. This is impossible since the eigenvalues of both the complete and approximate kernels are real.

In particular, on taking only the first term of the series in (16), we obtain

$$a_o = \frac{2iK_o}{(A_{oo} - \pi i)} \tag{17}$$

with $A_{oo} = 2 \sinh \alpha - \mathrm{Ei}(\alpha) + \mathrm{Ei}(-\alpha)$, and $K_o = \frac{1}{2}\left\{e^{-\alpha}\mathrm{Ei}(4\alpha) + e^{\alpha}\mathrm{Ei}(-4\alpha) - 2\cosh\alpha(\gamma + \ln 4\alpha)\right\}$.

The scattering coefficient is then

$$\frac{4K_o}{3(A_{oo}^2 + \pi^2)} .$$

We show in Table 1 some results obtained by using the approximation in which it is assumed that only a_o is non-zero. The approximate scattering coefficient is correct to within about 10% for $\alpha \leq 2$ but thereafter diverges rapidly.

Table 1

α	coefficient
0.5	0.00200
1.0	0.0387
1.5	0.253
2.0	0.884
2.5	1.617
3.0	2.039

One-half the scattering coefficient of a circular disc when $a_n = 0$ for $n \geq 1$.

The scattering coefficient obtained by assuming that only a_0 and a_1 are non-zero is given in Table 2. The values obtained from the approximation were calculated correct to 4 significant figures. Also shown are the correct values of Bouwkamp. It can be seen that the approximate values agree with Bouwkamp's to within 4% for $\alpha \leq 2.5$ (within $\frac{1}{2}$ % for $\alpha \leq 2$). (For an estimate of the error, independent of Bouwkamp's results, see Section 8 and Table 6.) The value at $\alpha = 3$ is 13% too high; this can be improved by taking more terms in the series. Thus if we take a_2 non-zero we find that the scattering coefficient is 2 x 1.158, which is correct to within 1 $\frac{1}{2}$ %.

Table 2

α	Correct	Approximation	α	Correct	Approximation
0.1	0.00000300	0.00000301	1.6	0.384	0.3819
0.2	0.0000486	0.00004865	1.7	0.505	0.5019
0.3	0.000250	0.0002503	1.8	0.642	0.6378
0.4	0.000809	0.0008089	1.9	0.783	0.7812
0.5	0.00203	0.002032	2.0	0.922	0.9210
0.6	0.00436	0.004363	2.1	1.042	1.046
0.7	0.00843	0.008422	2.2	1.135	1.148
0.8	0.0151	0.01506	2.3	1.200	1.224
0.9	0.0255	0.02541	2.4	1.235	1.274
1.0	0.0411	0.04100	2.5	1.248	1.3016
1.1	0.064	0.06380	2.6	1.244	1.313
1.2	0.097	0.09626	2.7	1.227	1.312
1.3	0.142	0.1413	2.8	1.202	1.303
1.4	0.203	0.2022	2.9	1.173	1.290
1.5	0.283	0.2815	3.0	1.142	1.272

The column headed 'Correct' is reproduced from Bouwkamp's review and gives the transmission coefficient of an aperture. The column headed 'Approximation' is one-half of the scattering coefficient of a disc determined by assuming that only a_0 and a_1 are non-zero. The value corresponding to $\alpha = 3$ when a_0, a_1 and a_2 are non-zero is 1.158.

One point that should be observed is that the approximation reproduces the maximum which occurs in the correct values although the magnitude is some 5% in error.

For small values of α the approximations (B13) - (B17) inclusive can be used. The corresponding approximations to a_0 and a_1 are

$$(18) \qquad \mathcal{R}(a_0) \approx -\frac{4\alpha^4}{\pi^2}\left(\frac{2}{9} + \frac{13\alpha^2}{225}\right), \quad \mathcal{I}(a_0) \approx \frac{16\alpha^7}{81\pi^3} + \frac{152\alpha^9}{2025\pi^3},$$

$$(19) \qquad \mathcal{R}(a_1) \approx -\frac{8\alpha^4}{15\pi^2}, \quad \mathcal{I}(a_1) \approx \frac{16\alpha^7}{135\pi^3}.$$

The scattering coefficient is twice

$$\frac{8\alpha^4}{27\pi^2}\left(1 + \frac{8}{25}\alpha^2\right)$$

approximately.

6. The symmetrical point source

We now consider what happens when the incident field is due to a point source on the axis of the disc. Suppose that the point source is at $(0,0,-z_0)$ where $z_0 > 0$; then

$$u_0(\rho_1,z_1) = \frac{\exp\left[-i\alpha\left\{\rho_1^2 + (z_1+z_0)^2\right\}^{1/2}\right]}{\left\{\rho_1^2 + (z_1 + z_0)^2\right\}^{1/2}}.$$

Since

$$\left(\frac{\partial u_o}{\partial z_1}\right)_{z_1=0} = \left(\frac{\partial u_o}{\partial z_o}\right)_{z_1=0}$$, we have

$$F(y) = \frac{\partial 4}{\partial z_o \pi}\int_{(1-y^2)^{1/2}}^{1} \frac{\cosh\left\{\alpha(t^2-1+y^2)^{\frac{1}{2}}\right\}}{(t^2-1+y^2)^{1/2}} \int_0^t \rho \frac{\cosh\left\{\alpha(t^2-\rho^2)^{\frac{1}{2}}\right\}}{(t^2-\rho^2)^{1/2}} \frac{\exp\left\{-i\alpha(\rho^2+z_o^2)^{\frac{1}{2}}\right\}}{(\rho^2+z_o^2)^{1/2}}\, d\rho dt$$

$$= -\frac{2i}{\pi}\frac{\partial}{\partial z_o}\int_{(1-y^2)^{1/2}}^{1} \frac{\cosh\left\{\alpha(t^2-1+y^2)^{1/2}\right\}}{(t^2-1+y^2)^{1/2}} \int_{-t-iz_o}^{t-iz_o} \frac{e^{\alpha u}}{u}\, du dt$$

or

$$F(y) = -\frac{4}{\pi} e^{-i\alpha z_o}\int_{(1-y^2)^{1/2}}^{1} \frac{\cosh\left\{\alpha(t^2-1+y^2)^{\frac{1}{2}}\right\}}{(t^2-1+y^2)^{1/2}}\, \frac{t\cosh \alpha t + iz_o \sinh \alpha t}{t^2 + z_o^2}\, dt. \tag{20}$$

By the same method that (9) was derived this may be converted into

$$F(y) = -\frac{2}{\pi} e^{-i\alpha z_o}\int_{1-y}^{1+y} \frac{2(t^2+1-y^2)\cosh \alpha t + 4itz_o \sinh \alpha t}{(t^2+1-y^2)^2 + 4t^2\, z_o^2}\, dt. \tag{21}$$

The integral equation for G may now be written down. The solution of this equation can be used to obtain the scattering coefficient, (which is defined as the ratio between the average rate at which energy is scattered and the average rate at which energy is incident on the disc); the result is

$$-\frac{(1+z_o^2)^{1/2}}{\alpha\left\{(1+z_o^2)^{1/2}-z_o\right\}} \mathcal{J}\int_0^1 g\, \frac{\partial}{\partial z_o} \frac{e^{i\alpha(\rho^2+z_o^2)^{1/2}}}{(\rho^2+z_o^2)^{1/2}}\, \rho\, d\rho. \tag{22}$$

(For a general proof see Jones[16].)

We shall now restrict our attention to the case when z_o is large. When $z_o \gg 1$, (20) shows that

$$F(y) = \frac{e^{-i\alpha z_o}}{z_o} \left\{ F_1(y) + \frac{1}{z_o} F_2(y) + \frac{1}{z_o^2} F_3(y) + O\left(\frac{1}{z_o^3}\right) \right\}$$

where

$$F_1(y) = -\frac{2i}{\pi} \int_{1-y}^{1+y} \frac{\sinh \alpha t}{t}\, dt,$$

$$F_2(y) = -\frac{1}{\pi\alpha} \left\{ \sinh \alpha(1+y) - \sinh \alpha(1-y) + \alpha(1+y)\cosh \alpha(1-y) - \alpha(1-y)\cosh \alpha(1+y) + \alpha^2(1-y^2) \int_{1-y}^{1+y} \frac{\sinh \alpha t}{t}\, dt \right\}$$

and

$$F_3(y) = \frac{i}{2\pi} \Bigg[(1+y) \left\{ \frac{1}{\alpha} - \frac{1}{2}\alpha(1-y)^2 \right\} \cosh \alpha(1+y) - \left\{ \frac{1}{\alpha^2} + \frac{1}{2}(1-y)^2 \right\} \sinh \alpha(1+y)$$

$$-(1-y) \left\{ \frac{1}{\alpha} - \frac{1}{2}\alpha(1+y)^2 \right\} \cosh \alpha(1-y) + \left\{ \frac{1}{\alpha^2} + \frac{1}{2}(1+y)^2 \right\} \sinh \alpha(1-y)$$

$$+2(1-y^2)\left\{ 1 + \frac{1}{4}\alpha^2(1-y^2) \right\} \int_{1-y}^{1+y} \frac{\sinh \alpha t}{t}\, dt \Bigg].$$

The reason for including the term involving F_3 will be given below.

Let

$$G(y) = \frac{e^{-i\alpha z_o}}{z_o} \left\{ G_1(y) + \frac{1}{z_o} G_2(y) + \frac{1}{z_o^2} G_3(y) + O\left(\frac{1}{z_o^3}\right) \right\}.$$

Then, since $F(y)$ is odd, the equation (10) may be transformed into the form (14); hence, taking corresponding powers of $\frac{1}{z_o}$ together, we obtain

$$(23) \qquad G_n(y) = F_n(y) + \frac{i}{\pi} \int_{-1}^{1} G_n(x) \frac{\sinh \alpha(x-y)}{x-y}\, dx \qquad (-1 \leq y \leq 1)$$

with $n = 1, 2$ or 3.

In view of the expression for F_1, G_1 satisfies (14), i.e., it is the solution for the incident plane wave. With regard to G_2 and G_3, we put

$$G_2(y) = F_2(y) + \sum_{n=o}^{\infty} \frac{b_n \alpha^{2n} y^{2n+1}}{(2n+1)!}$$

and

$$G_3(y) = F_3(y) + \sum_{n=o}^{\infty} \frac{c_n \alpha^{2n} y^{2n+1}}{(2n+1)!} .$$

The equations for b_n and c_n, analogous to (16), are

$$(24) \qquad \frac{\pi i b_m \alpha^{2m}}{(2m+1)!} = L_m + \sum_{n=o}^{\infty} A_{mn} b_n \qquad (m = 0, 1, \ldots)$$

and

$$(25) \qquad \frac{\pi i c_m \alpha^{2m}}{(2m+1)!} = M_m + \sum_{n=o}^{\infty} A_{mn} c_n \qquad (m = 0, 1, \ldots)$$

where

$$L_m = \frac{1}{(2m+1)!} \int_{-1}^{1} F_2(x) \frac{d^{2m+1}}{dx^{2m+1}} \frac{\sinh ax}{x} dx$$

and

$$M_m = \frac{1}{(2m+1)!} \int_{-1}^{1} F_3(x) \frac{d^{2m+1}}{dx^{2m+1}} \frac{\sinh ax}{x} dx .$$

The expression for the scattering coefficient may also be approximated when $z_o >> 1$. In order to remove an exponential from the integration we shall make the additional assumption that $\alpha/2z_o << 1$. Then the scattering coefficient is

$$- 2\mathcal{R}\int_0^1 G_1 y\,dy - \frac{2}{\alpha z_0}\mathcal{I}\int_0^1 \left[i\alpha G_2 - \left\{1 + \frac{1}{2}\alpha^2(1-y^2)\right\} G_1\right] y\,dy$$

(26)

$$- \frac{2}{\alpha z_0^2}\mathcal{I}\int_0^1 \left[i\alpha G_3 + \frac{3}{4} i\alpha G_1 - \left\{1 + \frac{1}{2}\alpha^2(1-y^2)\right\} G_2 - i\alpha\left\{\frac{3}{2}(1-y^2) + \frac{\alpha^2}{8}(1-y^2)^2\right\} G_1\right] y\,dy$$

$$+ O\left(\frac{1}{z_0^3}\right) .$$

The first term is the scattering coefficient for a normally incident plane wave. We shall now prove that the second term is identically zero, so that there is no correction to the plane wave scattering coefficient if the term F_3 in F is neglected. In fact, it will be shown that the scattering coefficient contains only even powers of $1/z_0$.

The integral equation (1) can be written as

$$\left[\frac{\partial u_0}{\partial z_1}\right] = TG \qquad (27)$$

where T is a linear operator. In (27) the brackets indicate the value at $z_1 = 0$ and we imagine that $\partial u_0/\partial z$ is expressed in terms of $1-\rho^2$. We may then write the integral in the scattering coefficient (22) as

$$\int_0^1 G\left[\frac{\partial u_0^*}{\partial z_1}\right] y\,dy = \left(G, \left[\frac{\partial u_0^*}{\partial z_1}\right]\right) \qquad \text{(say)},$$

where the star indicates a complex conjugate. Clearly $\left(G, \left[\frac{\partial u_0}{\partial z}\right]\right) = \left(\left[\frac{\partial u_0}{\partial z}\right], G\right)$. Suppose now that, when $z_0 \gg 1$ and $\alpha/2z_0 \ll 1$,

$$\left[\frac{\partial u_o}{\partial z_1}\right] = \left\{ \sum_{n=1}^{N-1} v_n/z_o^n + O(1/z_o^N) \right\} e^{-i\alpha z_o}$$

and

$$G = \left\{ \sum_{n=1}^{N-1} G_n/z_o^n + O(1/z_o^N) \right\} e^{-i\alpha z_o}$$

where v_n and G_n are independent of z_o. Then, from (27),

$$v_n = TG_n$$

and the integral I_1 in the scattering coefficient is given by

$$I_1 = \left(G, \left[\frac{\partial u_o^*}{\partial z_1}\right] \right) = \sum_{n=2}^{N} \frac{1}{z_o^n} \sum_{p=1}^{n-1} (G_p, v_{n-p}^*) + O(1/z_o^{N+1}).$$

Now v_{2p} is purely real and v_{2p+1} is purely imaginary. Hence

$$\begin{aligned}
\sum_{p=1}^{n-1} (G_p, v_{n-p}^*) &= \sum_{p=1}^{n-1} \delta_{n-p}(G_p, v_{n-p}) \\
&= \sum_{p=1}^{n-1} \delta_{n-p}(G_p, TG_{n-p}) \\
&= \sum_{p=1}^{n-1} \delta_{n-p}(TG_p, G_{n-p}) \\
&= \sum_{p=1}^{n-1} \delta_{n-p}(v_p, G_{n-p}) \\
&= \sum_{p=1}^{n-1} \delta_p\, \delta_{n-p}(G_p, v_{n-p}^*)
\end{aligned}$$

where $\delta_p = \pm 1$ according as p is even or odd. Since $\delta_p\, \delta_{n-p} = \pm 1$ according as n is even or odd we see that

$$I_1 = \sum_{n=1} \frac{1}{z_o^{2n}} \sum_{p=o}^{2n-1} (G_p, v_{2n-p}^*) .$$

Further, the factor of I_1 in (22) involves only even powers of z_0 and hence the scattering coefficient contains only even powers of $1/z_0$. This proves our earlier assertion.

The third term in (26), therefore, is the first which provides a correction to the plane wave scattering coefficient. This is the reason why it is necessary to include the term F_3 in the expansion of F in inverse powers of z_0. The scattering coefficient for the point source when $z_0 \gg 1$ and $\alpha/2z_0 \ll 1$ is the plane wave scattering coefficient together with

$$-\frac{2}{z_0^2}\mathcal{I}\sum_{n=0}^{\infty}\frac{\alpha^{2n}}{(2n+1)!(2n+3)}\left[i\,c_n-\left(1+\frac{\alpha^2}{2n+5}\right)\frac{b_n}{\alpha}+i\left\{\frac{3}{4}-\frac{\alpha^2+3(2n+7)}{(2n+5)(2n+7)}\right\}a_n\right]+O(1/z_0^4).$$

There is, of course, no contribution from the terms involving F_2 and F_3 in the expressions for G_2 and G_3 since these are derived from fields which do not radiate.

Instead of solving (24) and (25) we can find G_2 by means of iteration when α satisfies (15). The power series expansion for G_2 after two iterations is

$$\begin{aligned}G_2 = &-\frac{2y}{\pi}\left\{2+\alpha^2-\frac{4i}{9\pi}\alpha^3+\frac{\alpha^4}{12}-\frac{46i\alpha^5}{225\pi}+\left(\frac{1}{360}-\frac{8}{81\pi^2}\right)\alpha^6\right.\\&\qquad\left.-\frac{1598i}{44100}\alpha^7+\left(\frac{1}{18940}-\frac{96}{2025\pi^2}\right)\alpha^8+\dots\right\}\\&+\frac{2\alpha^2y^3}{3\pi}\left\{1-\frac{\alpha^2}{6}+\frac{2i\alpha^3}{15}-\frac{\alpha^4}{40}+\frac{11i\alpha^5}{175}-\left(\frac{1}{1008}-\frac{4}{135\pi^2}\right)\alpha^6+\dots\right\}\\&+\frac{2\alpha^4y^5}{180\pi}\left\{1-\frac{3\alpha^2}{10}+\frac{2i\alpha^3}{7}-\frac{3\alpha^4}{56}+\dots\right\}\\&+\frac{2\alpha^6y^7}{12600\pi}\left\{1-\frac{5\alpha^2}{14}+\dots\right\}+\frac{2\alpha^8y^9}{181440\pi}\left\{1+\dots\right\}.\end{aligned}$$

We can also solve (24) and (25) approximately in the same way as we solved (16) approximately, i.e., we take all b's and c's zero except b_0, b_1, c_0 and c_1.

The corresponding values of the scattering coefficient correction are given in Table 3. The expressions for L_o, L_1, M_o and M_1, necessary for this calculation are given by equations (B9) - (B12) inclusive.

α	Coefficient correction	α	Coefficient correction
0.1	0.0003010	1.6	0.1262
0.2	0.001214	1.7	0.1427
0.3	0.002768	1.8	0.1565
0.4	0.005014	1.9	0.1656
0.5	0.008025	2.0	0.1686
0.6	0.01190	2.1	0.1650
0.7	0.01675	2.2	0.1555
0.8	0.02274	2.3	0.1412
0.9	0.03002	2.4	0.1235
1.0	0.03880	2.5	0.1069
1.1	0.04925	2.6	0.08970
1.2	0.06151	2.7	0.07431
1.3	0.07571	2.8	0.06198
1.4	0.09151	2.9	0.05431
1.5	0.1087	3.0	0.05208

Table 3

The column headed 'Coefficient correction' gives $z_o^2/2$ times the term which must be added to the plane wave scattering coefficient to obtain the scattering coefficient for a point source such that $z_o \gg 1$ and $\alpha/2z_o \ll 1$ when it is assumed that only a_o, a_1, b_o, b_1, c_o and c_1 are non-zero.

When α is small we can make use of equations (B18) - (B21). The corresponding values of b_o, b_1, c_o and c_1 are

$$\mathcal{R}(b_o) \approx \frac{16\alpha^6}{81\pi^3} + \frac{232\alpha^8}{2025\pi^3}, \qquad \mathcal{I}(b_o) \approx \frac{8\alpha^3}{9\pi^2}\left(1 + \frac{23\alpha^2}{50}\right),$$

$$\mathcal{R}(b_1) \approx \frac{16\alpha^6}{135\pi^3}, \quad \mathcal{I}(b_1) \approx \frac{8\alpha^3}{15\pi^2},$$

$$\mathcal{R}(c_o) \approx \frac{4\alpha^4}{15\pi^2}\left(2 + \frac{61\alpha^2}{105}\right), \quad \mathcal{I}(c_o) \approx -\frac{4\alpha^7}{15\pi^3}\left(\frac{4}{9} + \frac{736\alpha^2}{475}\right),$$

$$\mathcal{R}(c_1) \approx \frac{8\alpha^4}{25\pi^2}, \quad \mathcal{I}(c_1) \approx -\frac{16\alpha^7}{225\pi^3}.$$

These results, together with (18) and (19), show that the correction to the plane wave scattering coefficient is approximately

$$\frac{2}{z_o^2}\left(\frac{8\alpha^2}{27\pi^2} + \frac{2\alpha^4}{25}\right).$$

7. An upper bound for the scattered amplitude

In this section we give the general theory for deriving an upper bound for the scattered amplitude and scattering coefficient in any scattering problem in which the non-radiating problem can be solved. Since the non-radiating picture is appropriate for low frequencies it may be anticipated that the upper bound will be most valuable at low frequencies. As an illustration rough upper bounds for the scattered amplitude and scattering coefficient of a circular disc in a normally incident plane wave will be determined and in the field due to a point source.

The mathematical problem in general scattering theory (for sound waves, electromagnetic waves and beams of atomic particles) may be presented as:

Find the solution g of

$$f = Tg \tag{28}$$

where f is known and T is a linear operator.†

The scattered field can be written as

$$[h,g] = [g,h]$$

where h is known and the brackets indicate an inner product, the square brackets being used to allow of its being distinguished from the inner product in Section 6. Illustrations of how particular problems can be written in this way can be found elsewhere (Jones[4]).

Let $T = T_r + iT_i$ where T_r and T_i are real linear operators, i.e., they produce real functions from real functions. Suppose that T_r^{-1}, the inverse of T_r, exists. The finding of T_r^{-1} corresponds to solving the non-radiating problems for the circular disc. The application of T_r^{-1} to (28) supplies

$$T_r^{-1} f = (1 + iS)\, g \tag{29}$$

where $S \equiv T_r^{-1} T_i$. Observe that S is a real operator.

Define the norm $||g||$ by the equation

$$||g|| = [g, g^*]^{1/2}$$

where the asterisk indicates a complex conjugate. Then

$$||(1 + iS)g||^2 = \left[(1 + iS)g,\ (1 - iS)g^*\right] = ||g||^2 + ||Sg||^2 \tag{30}$$

provided that S is a symmetric operator, i.e., provided that

$$\left[Sh_1,\ h_2\right] = \left[h_1,\ Sh_2\right].$$

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† The analysis is applicable to vector fields with suitable interpretation of f, g and T.

It follows from (30) that

$$\|g\|^2 \leq \|(1 + iS)g\|^2 .$$

Hence (29) shows that

$$(31) \qquad \|g\|^2 \leq \|T_r^{-1} f\|^2 .$$

Therefore

$$(32) \qquad \big|\, [h,g] \,\big| \leq \|h\| \;\|g\| \leq \|h\| \;\|T_r^{-1}f\|.$$

The inequalities (31) and (32) provide upper bounds for the norm of g and the scattered amplitude respectively when the operator $T_r^{-1}T_i$ is symmetric.

A related inequality which may be obtained in a similar way is

$$(33) \qquad \Big|\, \big[h,g - T_r^{-1}f\big] \,\Big| \leq \|h\| \;\; \|T_r^{-1}f\|.$$

A modification of (32) is useful in certain circumstances such as the calculation of the scattering coefficient. Here one may require $\mathcal{R}\,[h,g]$ (say) and know that

$$(34) \qquad \mathcal{R}\Big[h,\; T_r^{-1}f\Big] = 0.$$

The upper bound given by (32) is then usually much too high. To overcome this difficulty, put $g = T_r^{-1}f + g_1$, where

$$- iS\; T_r^{-1}f = (1 + iS)g_1 .$$

Then

$$\mathcal{R}\,[h,g] = \mathcal{R}\,[h,g_1]$$

on account of (34) and, therefore,

$$(35) \qquad |\mathcal{R}[h,g]| \leq |[h,g_1]| \leq ||h||\ ||ST_r^{-1}f||.$$

In general this is a better upper bound for $\mathcal{R}[h,g]$ than that supplied by (33). This real part may be replaced by the imaginary part without affecting the analysis provided that the analogue of (34) holds.

As an application of the above inequalities we obtain a rough upper bound for the field scattered by a circular disc in a normally incident plane wave. Equations (1) and (13) may then be identified with (28) and (29). The inner product is defined by

$$[h,g] = \int_0^1 h(\rho)\ g(\rho)\ \rho\ d\rho = \int_0^1 H(y)\ G(y)\ y\ dy$$

where $y^2 = 1 - \rho^2$ and the capital letters indicate functions obtained by the change of variable. The second form enables us to make the identification

$$T_r^{-1}f = -\frac{2i}{\pi}\int_{1-y}^{1+y} \frac{\sinh \alpha x}{x}\ dx$$

and

$$Sg = -\frac{1}{\pi}\int_0^1 \left\{ \frac{\sinh \alpha(x-y)}{x-y} - \frac{\sinh \alpha(x+y)}{x+y} \right\} G(x)\ dx.$$

Obviously S is symmetric and therefore formulas (31), (32) and (35) are available. Since

$$(36) \qquad \frac{\sinh \alpha x}{x} \leq \alpha \cosh \alpha x,$$

$$(37) \qquad |T_r^{-1}f| \leq \frac{4}{\pi} \cosh \alpha \sinh \alpha y .$$

Consequently, (31) implies that

$$\int_0^1 y|G(y)|^2 dy \leq \frac{4}{\pi^2\alpha^2} \cosh^2 \alpha\ R(\alpha)$$

where $R(\alpha) = \alpha(\sinh 2\alpha - 2\alpha) - \frac{1}{2}(\cosh 2\alpha - 1 - 2\alpha^2)$.

The scattered amplitude at infinity in the direction of the unit vector $\underline{n}$ may be taken as $\left[e^{i\alpha\underline{\rho}\cdot\underline{n}}, g\right]$. Therefore, inequalities (32) and (37) show that the modulus of the scattered amplitude is never greater than

$$\frac{\sqrt{2R}}{\pi\alpha}\cosh\alpha\,. \tag{38}$$

To find an upper bound for the scattering coefficient we observe that the scattering coefficient is $-2\mathcal{R}[1,g]$ and that $\mathcal{R}[1, T_r^{-1}f] = 0$. Thus equation (34) is satisfied and we may employ the upper bound given by (35). Now

$$0 \leq \frac{\sinh\alpha(x+y)}{x+y} - \frac{\sinh\alpha(x-y)}{x-y} \leq \frac{\alpha}{3}\left\{\cosh\alpha(x+y) - \cosh\alpha(x-y)\right\} \tag{39}$$

$$\leq \frac{2\alpha}{3}\sinh\alpha x\sinh\alpha y.$$

Therefore

$$|ST_r^{-1}f| \leq \frac{4\alpha}{3\pi^2}\int_0^1 \sinh\alpha x\sinh\alpha y\int_{1-x}^{1+x}\frac{\sinh\alpha t}{t}\,dt\,dx \leq A_1\sinh\alpha y,$$

where

$$A_1 = \frac{4\alpha}{3\pi^2}\int_0^1 \sinh\alpha x\int_{1-x}^{1+x}\frac{\sinh\alpha t}{t}\,dt\,dx$$

$$= \frac{1}{3\pi^2}\left[2\cosh\alpha\left\{Ei(2\alpha) - Ei(-2\alpha)\right\} - e^{-\alpha}Ei(4\alpha) + e^{\alpha}Ei(-4\alpha) - 2(\gamma + \ln 4\alpha)\sinh\alpha\right].$$

Hence, from (35),

$$2\left|\mathcal{R}[1,g]\right| \leq \frac{A_1}{\alpha}\left(\frac{1}{2}R\right)^{1/2}. \tag{40}$$

Some values of the upper bound for the scattering coefficient, as given by (40), are shown in Table 4. It can be seen that the bound gives an indication of

α	Upper bound	α	Upper bound
0.1	0.00000319	1.1	0.0763
0.2	0.0000518	1.2	0.119
0.3	0.000267	1.3	0.181
0.4	0.000870	1.4	0.271
0.5	0.00220	1.5	0.400
0.6	0.00478	1.6	0.586
0.7	0.00934	1.7	0.851
0.8	0.0169	1.8	1.228
0.9	0.0291	1.9	1.761
1.0	0.0479	2.0	2.515

Table 4

The column headed 'Upper bound' shows one-half of the right hand side of (40). It gives therefore, an upper bound for the transmission coefficient of a circular aperture.

the order of magnitude of the coefficient and is within 10% of the correct value for $\alpha \leq 0.7$ and within 50% for $\alpha < 1.6$. The estimate (and that for the scattered amplitude) could be improved to some extent by replacing the rough inequalities (36) and (39) by more refined ones.

To obtain an upper bound for the scattering coefficient when the incident wave is supplied by a point source we make the identification

$$T^{-1} f = F$$

where F is given by (21). The scattering coefficient is, from (22),

$$-\frac{(1+z_o^2)^{1/2}}{\alpha\left\{(1+z_o^2)^{1/2}-z_o\right\}} \mathcal{I}\left[G, \frac{\partial}{\partial z_o} \frac{e^{i\alpha(1-y^2+z_o^2)^{1/2}}}{(1-y^2+z_o^2)^{1/2}}\right]$$

where the inner product is defined in the same way as for the incident plane wave. Also

$$\mathscr{I}\left[F, \frac{\partial}{\partial z_0} \frac{e^{i\alpha(1-y^2+z_0^2)^{1/2}}}{(1-y^2+z_0^2)^{1/2}}\right] = 0,$$

so that we can use (35) with $\mathscr{R}$ replaced by $\mathscr{I}$. Hence the scattering coefficient is never greater than

$$\frac{(1+z_0^2)^{1/2}}{\alpha\left\{(1+z_0^2)^{1/2}-z_0\right\}} \left\| \frac{\partial}{\partial z_0} \frac{e^{i\alpha(1-y^2+z_0^2)^{1/2}}}{(1-y^2+z_0^2)^{1/2}} \right\| \, \|SF\|.$$

Now, a use of (39) gives

$$|SF| \leq B \sinh \alpha y$$

where

$$B^2 = \left\{ \frac{8\alpha}{3\pi^2} \int_0^1 \sinh \alpha x \int_{1-x}^{1+x} \frac{(t^2+1-x^2)\cosh \alpha t}{(t^2+1-x^2)^2+4t^2z_0^2} \, dt \, dx \right\}^2$$

(41)

$$+ \left\{ \frac{16\alpha z_0}{3\pi^2} \int_0^1 \sinh \alpha x \int_{1-x}^{1+x} \frac{t \sinh \alpha t}{(t^2+1-x^2)^2+4t^2z_0^2} \, dt \, dx \right\}^2 .$$

Hence

(42) $$\|SF\| \leq \frac{B}{2\alpha} R^{1/2} .$$

Also

$$\left\| \frac{\partial}{\partial z_0} \frac{e^{i\alpha(1-y^2+z_0^2)^{1/2}}}{(1-y^2+z_0^2)^{1/2}} \right\|^2 = z_0^2 \int_0^1 y \left\{ \frac{\alpha^2}{(1-y^2+z_0^2)^2} + \frac{1}{(1-y^2+z_0^2)^3} \right\} dy$$

$$= \frac{1}{4(1+z_0^2)^2} \left\{ 2\alpha^2(1+z_0^2) + 2 + \frac{1}{z_0^2} \right\} .$$

Combining this result with (42) we find that the scattering coefficient can never exceed

$$\frac{\left\{2\alpha^2(1+z_o^2) + 2 + \frac{1}{z_o^2}\right\}^{1/2} BR^{1/2}}{4\alpha^2(1+z_o^2)^{1/2}\left\{(1+z_o^2)^{1/2} - z_o\right\}} .$$

The actual evaluation of B from (41) is not easy but there are several ways of finding an upper bound for B. One of these will now be described. We have

$$(43)\ \frac{16\alpha z_o}{3\pi^2}\int_0^1 \sinh\alpha x \int_{1-x}^{1+x} \frac{t \sinh \alpha t}{(t^2+1-x^2)^2+4t^2z_o^2}\, dtdx \le \frac{4\alpha}{3\pi^2 z_o}\int_0^1 \sinh\alpha x \int_{1-x}^{1+x} \frac{\sinh\alpha t}{t}\, dtdx \le A_1/z_o$$

and

$$\frac{8\alpha}{3\pi^2}\int_0^1 \sinh\alpha x \int_{1-x}^{1+x} \frac{(t^2+1-x^2)\cosh\alpha t}{(t^2+1-x^2)^2+4t^2z_o^2}\, dtdx \le \frac{2\alpha}{3\pi^2 z_o^2}\int_0^1 \sinh\alpha x \int_{1-x}^{1+x} \left(1+ \frac{1-x^2}{t^2}\right)\cosh\alpha t\, dtdx$$

$$(44) \qquad \le B_1/z_o^2$$

where

$$B_1 = \frac{2\alpha}{3\pi^2}\int_0^1 \sinh \alpha x \int_{1-x}^{1+x} \left(1 + \frac{1-x^2}{t^2}\right) \cosh \alpha t\, dt\, dx$$

$$= \frac{1}{3\pi^2}\Big[(1+ \frac{1}{\alpha})e^{-\alpha}Ei(4\alpha)+(1- \frac{1}{\alpha})e^{\alpha}Ei(-4\alpha)-2(\cosh \alpha- \frac{\sinh \alpha}{\alpha})(\gamma + \ln 4\alpha)$$

$$+ 2(\sinh \alpha- \frac{\cosh \alpha}{\alpha}) \left\{Ei(2\alpha)-Ei(-2\alpha)\right\} -4(\frac{\sinh 2\alpha}{2\alpha} -1) \cosh \alpha\Big] .$$

Thus the scattering coefficient can never exceed

$$(45)\qquad \frac{1}{\alpha}\left\{\alpha^2(1+z_o^2)+1+\frac{1}{2z_o^2}\right\}^{\frac{1}{2}}\left\{1+\frac{z_o}{(1+z_o^2)^{1/2}}\right\}\left\{\frac{A_1^2\,R}{8\alpha^2 z_o^2}+\frac{B_1^2 R}{8\alpha^2 z_o^4}\right\}^{\frac{1}{2}} .$$

The quantity $A_1(R/8\alpha^2)^{1/2}$ is tabulated in Table 4 and $B_1(R/8\alpha^2)^{1/2}$ is given in Table 5 so that the upper bound for the scattering coefficient given by (45) can be computed for a given z_o without great difficulty.

α	$B_1(R/8\alpha^2)^{1/2}$	α	$B_1(R/8\alpha^2)^{1/2}$
0.1	0.0000320	1.1	0.0857
0.2	0.000261	1.2	0.126
0.3	0.000908	1.3	0.184
0.4	0.00225	1.4	0.266
0.5	0.00463	1.5	0.382
0.6	0.00854	1.6	0.545
0.7	0.0146	1.7	0.774
0.8	0.0239	1.8	1.097
0.9	0.0375	1.9	1.551
1.0	0.0573	2.0	2.191

Table 5

With regard to the actual value of the scattering coefficient we may note that, if B were calculated exactly by (41), we would expect the upper bound obtained to be in excess of the correct value by a percentage of the same order as that by which the upper bound in Table 4 exceeds the correct value, because the same method is used in both estimates. The upper bound given by (45) is still further in excess because of the use of the inequalities (43) and (44). It is easy to see that the additional increase is less than $100/(1+z_o^2)\%$ of the value given by (47). In particular, if $z_o \geq 10$ the increase is less than 1% and if $z_o^2 \geq 10$ the increase is less than 10%. Consequently, it is suggested that an estimate of the scattering coefficient can be obtained by taking the same percentage of (45) that has to be taken of the right-hand side of (40) to obtain the correct value for the plane wave scattering coefficient. This estimate should not be in error by more than 1% (of itself) for $z_o \geq 10$ and should differ from the correct value by not more than 10% (of itself) for $z_o^2 \geq 10$.

8. An estimate of the error introduced by approximation of the kernel

Since we have replaced the original integral equation of the first kind (1) by an integral equation of the second kind (10) we are in a much stronger position than before for estimating the error made by approximate methods. For example, when we solve (10) approximately by approximation to the kernel as in Sections 5 and 6 we can rewrite the exact equation so that one term involves the difference between the exact and approximate kernels. When this difference is not too large, the value of this term can be determined by iteration, and the exact equation becomes an integral equation which can be solved by solving a finite number of simultaneous equations. The difference between the exact solution and the approximate solution may therefore be estimated from a knowledge of the magnitude of the difference between the exact and approximate kernels and a knowledge of the inverse of the approximate kernel.

Other processes can be devised, and the one that we employ takes advantage of the results proved in the preceding section. Attention will be restricted to the problem in which the incident wave is plane.

Equation (13) can be written as

$$(1 + iS)G = F. \tag{46}$$

Define the inner product $[H,G]$ by

$$[H,G] = \int_0^1 H(y)G(y)dy,$$

so that the scattering coefficient for an incident plane wave is $-2\mathcal{R}[y,G]$.[†] When equation (46) is solved approximately by approximation to the kernel we solve

$$(1 + iS_o)G_o = F \tag{47}$$

and take the scattering coefficient to be $-2\mathcal{R}[y,G_o]$.

Let $G = G_o + G_1$. Then the error made in the scattering coefficient by our approximation is $-2\mathcal{R}[y,G_1]$. The modulus of the error does not exceed $2\,||y||\ ||G_1||$, i.e., $2\,||G_1||\,/\sqrt{3}$.

Now

$$(1 + iS)G_1 = (1 + iS)\,(G-G_o) = F - (1 + iS)G_o$$

or

$$(1 + iS)G_1 = -i(S - S_o)G_o\,, \tag{48}$$

from (47). Equation (48) is of the same type as (29) and S is symmetric. Hence an inequality of the type (31) holds, namely

$$||G_1|| \leq ||(S - S_o)G_o||.$$

Hence the modulus of the error made in the scattering coefficient does not exceed $2\,||(S - S_o)G_o||\,/\sqrt{3}$. Since all quantities are known in this formula we have an explicit expression for a bound to the error.

[†] Only slight modification of the following analysis is required if y is replaced by the general function H. Thus, effectively, we have a method for estimating the error in calculating the scattering coefficient with an arbitrary incident field and in calculating the scattering field at any angle from the solution of the integral equation with the approximate kernel. Note also that the method assesses the accuracy of the approximate pressure distribution, since it involves an estimate of the norm of the difference between the exact and approximate distributions.

A reasonable estimate for $||(S - S_o)G_o||$ may be obtained as follows. We take

$$S_oG = \frac{2}{\pi} \sum_{m=o}^{N-1} \frac{y^{2m+1}}{(2m+1)!} \int_o^1 G(x) \frac{d^{2m+1}}{dx^{2m+1}} \frac{\sinh \alpha x}{x} dx .$$

Then, if $H(x) \geq 0$ for $0 \leq x \leq 1$,

$$0 \leq (S-S_o)H = \frac{2}{\pi} \sum_{m=N}^{\infty} \frac{y^{2m+1}}{(2m+1)!} \int_o^1 H(x) \frac{d^{2m+1}}{dx^{2m+1}} \frac{\sinh \alpha x}{x} dx .$$

Also

$$\frac{d^{2m+1}}{dx^{2m+1}} \frac{\sinh \alpha x}{x} = \sum_{s=0}^{\infty} \frac{\alpha^{2s+2m+3} x^{2s+1}}{(2s+1)! (2s+2m+3)} \leq \frac{\alpha^{2m+2}}{2m+3} \sinh \alpha x.$$

Hence $0 \leq (S - S_o)H$

$$\leq \frac{2}{\pi} \sum_{m=N}^{\infty} \frac{y^{2m+1}\alpha^{2m+2}}{(2m+1)!(2m+3)} \int_o^1 H(x) \sinh \alpha x \, dx$$

(49)

$$\leq \frac{2\alpha P(y)}{\pi(2N+3)} \int_o^1 H(x) \sinh \alpha x \, dx ,$$

where

$$P(y) = \sinh \alpha y - \sum_{m=0}^{N-1} \alpha^{2m+1} y^{2m+1} \Big/ (2m+1)! .$$

Consider the special case in which $N = 2$; this is the value employed in Section 5. We have

$$G_o = - \frac{2i}{\pi} \int_{1-x}^{1+x} \frac{\sinh \alpha t}{t} dt + a_o x + \frac{a_1 \alpha^2 x^3}{6} .$$

Let $a_o = \alpha_o + i\beta_o$ and $a_1 = \alpha_1 + i\beta_1$, where α_o, α_1, β_o and β_1 are real. An inspection of the values obtained reveals that α_o and α_1 are negative whereas β_o and β_1 are positive for all values of $\alpha \leq 3$. Thus, for $\alpha \leq 3$, $\mathcal{R}G_o \leq 0$ when $0 \leq x \leq 1$. More care is required with the imaginary part of G_o since it involves two terms of opposite sign. However, it can be verified without difficulty that, for $\alpha \leq 2.4$, $\mathcal{I} G_o \leq 0$ for $0 \leq x \leq 1$. Hence, when $\alpha \leq 2.4$, (49) gives

$$(50) \qquad 0 \leq -(S-S_o)\,\mathcal{R}\, G_o \leq \frac{2\alpha}{7\pi} A_2 P(y)$$

and

$$(51) \qquad 0 \leq -(S-S_o)\,\mathcal{I}\, G_o \leq \frac{2\alpha}{7\pi} \left(\frac{3\pi}{2\alpha} A_1 - A_3\right) P(y)$$

where now $P(y) = \sinh \alpha y - \alpha y - \alpha^3 y^3/6$, and

$$A_2 = \int_0^1 (\alpha_o x + \alpha_1 \alpha^2 x^3/6) \sinh \alpha x \, dx$$

$$= \frac{1}{\alpha} \left\{ (\alpha_o + \alpha_1 + \alpha_1 \alpha^2/6) \cosh \alpha - (\alpha_o + \alpha_1 + \alpha_1 \alpha^2/2) \frac{\sinh \alpha}{\alpha} \right\}$$

and

$$A_3 = \frac{1}{\alpha} \left\{ (\beta_o + \beta_1 + \beta_1 \alpha^2/6) \cosh \alpha - (\beta_o + \beta_1 + \beta_1 \alpha^2/2) \frac{\sinh \alpha}{\alpha} \right\} .$$

Therefore

$$\|(S-S_o)G_o\| \leq \frac{2\alpha}{7\pi} \left\{ A_2^2 + \left(\frac{3\pi}{2\alpha} A_1 - A_3\right)^2 \right\}^{\frac{1}{2}} R_1^{\frac{1}{2}}$$

where

$$R_1 = \int_0^1 \left\{ P(y) \right\}^2 dy =$$

$$= \frac{1}{2} \left(\frac{\sinh 2\alpha}{2\alpha} - 1\right) - 2 \left\{ (2 + \frac{\alpha^2}{6}) \cosh \alpha - (\frac{2}{\alpha} + \frac{\alpha}{2}) \sinh \alpha \right\} + \frac{\alpha^2}{3} + \frac{\alpha^4}{15} + \frac{\alpha^6}{252} .$$

Consequently, when $\alpha \leq 2.4$, the error in the scattering coefficient made by our approximation is less, in magnitude, than

$$(52) \qquad \frac{4\alpha}{7\pi\sqrt{3}} \left\{ A_2^2 + \left(\frac{3\pi}{2\alpha} A_1 - A_3\right)^2 \right\}^{\frac{1}{2}} R_1^{\frac{1}{2}} .$$

The values obtained from this formula are displayed in Table 6. It can be seen that the method gives adequate estimates of the error. The method can, of course, be adopted to cover a G_o which changes sign by, for example, replacing G_o by its modulus or the maximum of its modulus in the derivation of inequalities similar to (50) and (51).

	Error bound		Error bound
0.1	$0.0^{12}54$	1.3	0.0009
0.2	$0.0^{9}14$	1.4	0.0015
0.3	$0.0^{8}28$	1.5	0.0028
0.4	$0.0^{7}38$	1.6	0.0050
0.5	$0.0^{6}24$	1.7	0.0086
0.6	0.0000011	1.8	0.015
0.7	0.0000037	1.9	0.024
0.8	0.000012	2.0	0.037
0.9	0.000023	2.1	0.057
1.0	0.000077	2.2	0.084
1.1	0.0002	2.3	0.12
1.2	0.0004	2.4	0.17

Table 6

The column headed 'Error bound' gives one-half of the quantity shown in (52). It therefore gives a bound to the error in the 'Approximation' column of Table 1.

Acknowledgment

I wish to express my appreciation of the computation carried out by Mr. J.E. Iehnsen and Mr. D. F. Ferguson.

Appendix A

In this appendix we consider, firstly, the solution of the integral equation

$$\text{(A1)} \qquad f(\rho_1) = \frac{1}{\pi\rho_1} \frac{d}{d\rho_1} \int_0^{\rho_1} h(t) \frac{\cos\left\{ \alpha(\rho_1^2 - t^2)^{1/2} \right\}}{(\rho_1^2 - t^2)^{1/2}} \, dt.$$

This equation may be rewritten as

$$\text{(A2)} \qquad f_1(y) = \int_0^y h_1(x) \frac{\cos \alpha(y-x)^{1/2}}{(y-x)^{1/2}} \, dx$$

where

$$f_1(y) = \int_0^{\sqrt{y}} \pi\rho \, f(\rho) d\rho$$

and

$$h_1(x) = h(\sqrt{x}\,) / 2\sqrt{x} \;.$$

Equation (A2) can be solved by means of the Heaviside operator p. Denoting operational representations by capital letters, we obtain

$$F_1(p) = (\pi/p)^{1/2} \; e^{-\alpha^2/4p} \; H_1(p)$$

or

$$H_1(p) = (p/\pi)^{1/2} \; e^{\alpha^2/4p} \; F_1(p).$$

Hence

$$\text{(A3)} \qquad h_1(x) = \frac{1}{\pi} \frac{d}{dx} \int_0^x f_1(y) \frac{\cosh \alpha(x-y)^{1/2}}{(x-y)^{1/2}} \, dy,$$

so that

$$\text{(A4)} \qquad h(x) = 2x \int_0^x f(y)\, y \frac{\cosh \alpha(x^2-y^2)^{1/2}}{(x^2-y^2)^{1/2}} \, dy.$$

Secondly, we solve the equation

$$\int_y^1 g'(x) \; \frac{\cos\left\{\alpha(x^2-y^2)^{1/2}\right\}}{(x^2-y^2)^{1/2}} \; dx = f(y).$$

This equation can be written

$$f_2(y) = \int_0^y g_2(x) \; \frac{\cos\left\{\alpha(y-x)^{1/2}\right\}}{(y-x)^{1/2}} \; dx$$

where

$$f_2(y) = f\left\{(1-y)^{1/2}\right\}$$

and

$$g_2(x) = \frac{g'\left\{(1-x)^{1/2}\right\}}{2(1-x)^{1/2}} \; .$$

This equation is of the same type as (A2) and hence its solution is given by (A3). Therefore

$$\frac{g'\left\{(1-x)^{1/2}\right\}}{2(1-x)^{1/2}} = \frac{1}{\pi}\frac{d}{dx}\int_0^x f\left\{(1-y)^{1/2}\right\} \frac{\cosh \alpha(x-y)^{1/2}}{(x-y)^{1/2}} \; dy$$

or

$$g'(x) = -\frac{2}{\pi}\frac{d}{dx}\int_x^1 f(y)\, y \; \frac{\cosh\left\{\alpha(y^2-x^2)^{1/2}\right\}}{(y^2-x^2)^{1/2}} \; dx.$$

In order that g(1) = 0 it is necessary that

$$g(x) = -\frac{2}{\pi}\int_x^1 f(y)\, y \; \frac{\cosh\left\{\alpha(y^2-x^2)^{1/2}\right\}}{(y^2-x^2)^{1/2}} \; dx \; . \tag{A5}$$

Appendix B

In this appendix we evaluate the constants required for the solution of the integral equations (14) and (23) by means of simultaneous equations. We have

$$A_{mn} = \frac{\alpha^{2n}}{(2m+1)!(2n+1)!} \int_{-1}^{1} x^{2n+1} \frac{d^{2m+1}}{dx^{2m+1}} \frac{\sinh \alpha x}{x} \, dx.$$

If we expand the integrand in powers of x we see at once that $A_{mn} = A_{nm}$. Now

$$\begin{aligned} A_{mn} &= \frac{\alpha^{2n}}{(2m+1)!(2n+1)!} \left\{ 2 \left[\frac{d^{2m}}{dx^{2m}} \frac{\sinh \alpha x}{x} \right]_{x=1} - (2n+1) \int_{-1}^{1} x^{2n} \frac{d^{2m}}{dx^{2m}} \frac{\sinh \alpha x}{x} \, dx \right\} \\ &= \frac{2\alpha^{2n}}{(2m+1)!(2n+1)!} \left[\frac{d^{2m}}{dx^{2m}} \frac{\sinh \alpha x}{x} \right]_{x=1} - \frac{\alpha}{2m+1} A_{m-1/2,n-1/2} \, . \end{aligned} \tag{B1}$$

Similarly

$$A_{nm} = \frac{2\alpha^{2m}}{(2m+1)!(2n+1)!} \left[\frac{d^{2n}}{dx^{2n}} \frac{\sinh \alpha x}{x} \right]_{x=1} - \frac{\alpha}{2n+1} A_{n-1/2,m-1/2} \, .$$

Since $A_{mn} = A_{nm}$ and $A_{m-1/2,n-1/2} = A_{n-1/2,m-1/2}$,

$$2(m-n)A_{mn} = \frac{2}{(2m+1)!(2n+1)!} \left\{ \alpha^{2n}(2m+1) \left[\frac{d^{2m}}{dx^{2m}} \frac{\sinh \alpha x}{x} \right]_{x=1} - \alpha^{2m}(2n+1) \left[\frac{d^{2n}}{dx^{2n}} \frac{\sinh \alpha x}{x} \right]_{x=1} \right\} \quad \text{if } m \neq n.$$

Now

$$\frac{d^{2m}}{dx^{2m}} \frac{\sinh \alpha x}{x} = (2m)! \sum_{s=0}^{m} \frac{\alpha^{2s}}{(2s)!} \frac{\sinh \alpha x}{x^{2m+1-2s}} - (2m)! \sum_{s=1}^{m} \frac{\alpha^{2s-1}}{(2s-1)!} \frac{\cosh \alpha x}{x^{2m+2-2s}} \, , \tag{B2}$$

the last term being absent for m = 0. Hence, when $m \neq n$,

$$A_{mn} = \frac{1}{(2m+1)!(2n+1)!(m-n)} \left[\left\{ (2m+1)! \sum_{s=0}^{m} \frac{\alpha^{2n+2s}}{(2s)!} - (2n+1)! \sum_{s=0}^{n} \frac{\alpha^{2m+2s}}{(2s)!} \right\} \sinh \alpha \right.$$

(B3)

$$\left. + \left\{ (2n+1)! \sum_{s=1}^{n} \frac{\alpha^{2m+2s-1}}{(2s-1)!} - (2m+1)! \sum_{s=1}^{m} \frac{\alpha^{2n+2s-1}}{(2s-1)!} \right\} \cosh \alpha \right],$$

the series starting at $s = 1$ being absent whenever the summation would involve $s = 0$.

When $m = n$, (B1) and (B2) give

$$A_{mm} = \frac{2\alpha^{2m}}{(2m+1)!(2m+1)} \left\{ \sum_{s=0}^{m} \frac{\alpha^{2s}}{(2s)!} \sinh \alpha - \sum_{s=1}^{m} \frac{\alpha^{2s-1}}{(2s-1)!} \cosh \alpha \right\} - \frac{\alpha}{2m+1} A_{m-1/2,m-1/2}.$$

Also, after using

$$\frac{d^{2m-1}}{dx^{2m-1}} \frac{\sinh \alpha x}{x} = (2m-1)! \sum_{s=1}^{m} \frac{\alpha^{2s-1}}{(2s-1)!} \frac{\cosh \alpha x}{x^{2m+1-2s}} - (2m-1)! \sum_{s=0}^{m-1} \frac{\alpha^{2s}}{(2s)!} \frac{\sinh \alpha x}{x^{2m-2s}},$$

we find

$$A_{m-1/2,m-1/2} = \frac{2\alpha^{2m-1}}{(2m)!2m} \left\{ \sum_{s=1}^{m} \frac{\alpha^{2s-1}}{(2s-1)!} \cosh \alpha - \sum_{s=0}^{m-1} \frac{\alpha^{2s}}{(2s)!} \sinh \alpha \right\} - \frac{\alpha}{2m} A_{m-1,m-1}.$$

Therefore

$$A_{mm} = \frac{2\alpha^{2m}}{(2m+1)!} \left[\frac{\alpha^{2m}}{(2m)!(2m+1)} \sinh \alpha + \sum_{s=0}^{m-1} \frac{\alpha^{2s}}{(2s)!} \left\{ \frac{1}{2m+1} + \frac{1}{2m} \right\} \sinh \alpha \right.$$

$$\left. - \sum_{s=1}^{m} \frac{\alpha^{2s-1}}{(2s-1)!} \left\{ \frac{1}{2m+1} + \frac{1}{2m} \right\} \cosh \alpha \right] + \frac{\alpha^{2}}{2m(2m+1)} A_{m-1,m-1}.$$

Repeating the process we obtain

$$A_{mm} = \frac{2\alpha^{2m}}{(2m+1)!} \left[\sum_{s=0}^{m} \frac{\alpha^{2s}}{(2s)!} \left\{ \Psi(2m+2) - \Psi(2s+1) \right\} \sinh \alpha - \sinh \alpha \right.$$

$$\left. - \sum_{s=1}^{m} \frac{\alpha^{2s-1}}{(2s-1)!} \left\{ \Psi(2m+2) - \Psi(2s) \right\} \cosh \alpha \right] + \frac{\alpha^{2m}}{(2m+1)!} A_{oo}.$$

Since

$$\text{(B4)}\quad A_{oo} = \int_{-1}^{1} x \frac{d}{dx} \frac{\sinh \alpha x}{x} dx = 2 \sinh \alpha - \mathrm{Ei}(\alpha) + \mathrm{Ei}(-\alpha),$$

$$\text{(B5)}\quad A_{mm} = \frac{2\alpha^{2m}}{(2m+1)!} \left[\sum_{s=0}^{m} \frac{\alpha^{2s}}{(2s)!} \left\{ \Psi(2m+2) - \Psi(2s+1) \right\} \sinh \alpha - \sum_{s=1}^{m} \frac{\alpha^{2s-1}}{(2s-1)!} \cdot \left\{ \Psi(2m+2) - \Psi(2s) \right\} \cosh \alpha - \frac{1}{2} \left\{ \mathrm{Ei}(\alpha) - \mathrm{Ei}(-\alpha) \right\} \right].$$

The evaluation of K_m for $m = 0$ and $m = 1$ is not difficult by direct means, but the best method for general results appears to be the following. We have

$$K_m = \frac{1}{(2m+1)!} \int_{-1}^{1} \int_{1-x}^{1+x} \frac{\sinh \alpha t}{t} \frac{d^{2m+1}}{dx^{2m+1}} \frac{\sinh \alpha x}{x} dt\, dx$$

$$= \left[\frac{d^{2m}}{dy^{2m}} \frac{1}{(2m+1)!} \int_{-1}^{1} \int_{1-x}^{1+x} \frac{\sinh \alpha t}{t} \frac{d}{dx} \frac{\sinh \alpha(x-y)}{x-y} dt\, dx \right]_{y=0}$$

$$= \frac{2}{(2m+1)!} \left[\frac{d^{2m}}{dy^{2m}} Y(y) \right]_{y=0}$$

where

$$Y(y) = \frac{\sinh \alpha(1-y)}{1-y} \int_{0}^{2} \frac{\sinh \alpha t}{t} dt - \int_{-1}^{1} \frac{\sinh \alpha(x-y)}{x-y} \frac{\sinh \alpha(1-x)}{1-x} dx.$$

A straightforward calculation shows that

$$-4(1-y)Y(y) = \left[\mathrm{Ei}\left\{2\alpha(1-y)\right\} - \mathrm{Ei}\left\{-2\alpha(1+y)\right\} - \mathrm{Ei}\left\{2\alpha(y-1)\right\} + \mathrm{Ei}\left\{2\alpha(1+y)\right\} + C \right] \cdot \sinh \alpha(1-y) - \left[\mathrm{Ei}\left\{2\alpha(1-y)\right\} - \mathrm{Ei}\left\{-2\alpha(1+y)\right\} + \mathrm{Ei}\left\{2\alpha(y-1)\right\} - \mathrm{Ei}\left\{2\alpha(1+y)\right\} + 2 \ln \frac{1+y}{1-y} + D \right] \cosh \alpha(1-y)$$

where

$$C = \mathrm{Ei}(4\alpha) - \mathrm{Ei}(-4\alpha) - 2\mathrm{Ei}(2\alpha) + 2\mathrm{Ei}(-2\alpha)$$

and

$$D = \mathrm{Ei}(4\alpha) + \mathrm{Ei}(-4\alpha) - 2\gamma - 2\ln 4\alpha .$$

Hence

$$\begin{aligned} K_m = \frac{2}{(2m+1)!} \Bigg[& - \sum_{r=1}^{m} \frac{(2m)!}{(2r)!(2m-2r)!} \frac{d^{2m-2r}}{dx^{2m-2r}} \frac{\sinh \alpha x}{x} \frac{d^{2r-1}}{dx^{2r-1}} \frac{\sinh 2\alpha x}{x} \\ & - \frac{1}{4} \left\{ \mathrm{Ei}(4\alpha) - \mathrm{Ei}(-4\alpha) \right\} \frac{d^{2m}}{dx^{2m}} \frac{\sinh \alpha x}{x} \\ & + \sum_{r=0}^{m-1} \frac{(2m)}{(2r+1)!(2m-2r-1)!} \frac{d^{2m-2r-1}}{dx^{2m-2r-1}} \frac{\cosh \alpha x}{x} \frac{d^{2r}}{dx^{2r}} \frac{\cosh 2\alpha x}{x} \\ & + \frac{1}{4} D \frac{d^{2m}}{dx^{2m}} \frac{\cosh \alpha x}{x} - \sum_{r=0}^{m-1} \frac{(2m)!}{(2m-2r-1)!(2r+1)} \frac{d^{2m-2r-1}}{dx^{2m-2r-1}} \frac{\cosh \alpha x}{x} \Bigg]_{x=1} . \end{aligned}$$

Typical values are

$$K_o = \frac{1}{2} e^{-\alpha}\mathrm{Ei}(4\alpha) + \frac{1}{2} e^{\alpha}\mathrm{Ei}(-4\alpha) - (\gamma + \ln 4\alpha)\cosh \alpha, \tag{B6}$$

$$\begin{aligned} K_1 = \frac{1}{12} \Big[& (\alpha^2+2\alpha+2)e^{-\alpha}\mathrm{Ei}(4\alpha) + (\alpha^2-2\alpha+2)e^{\alpha}\mathrm{Ei}(-4\alpha) - 4\cosh \alpha(\cosh 2\alpha-1) \\ & - 8\alpha \sinh \alpha - 2 \left\{ (\alpha^2+2)\cosh \alpha - 2\alpha \sinh \alpha \right\} (\gamma + \ln 4\alpha) \Big] , \end{aligned} \tag{B7}$$

$$\begin{aligned} K_2 = \frac{1}{240} \Big[& (\alpha^4+4\alpha^3+12\alpha^2+24\alpha+24)e^{-\alpha}\mathrm{Ei}(4\alpha) + (\alpha^4-4\alpha^3+12\alpha^2-24\alpha+24)e^{\alpha}\mathrm{Ei}(-4\alpha) \\ & - 2 \left\{ (\alpha^4+12\alpha^2+24)\cosh \alpha - 4\alpha(\alpha^2+6)\sinh \alpha \right\} (\gamma + \ln 4\alpha) \\ & - 8(\alpha^2+7)\cosh \alpha \cosh 2\alpha - 16\alpha(\alpha^2+7)\sinh \alpha + 8(5\alpha^2+7)\cosh \alpha \Big] . \end{aligned} \tag{B8}$$

With regard to the evaluation of L_m, no attempt has been made to provide a general expression. Instead L_o and L_1 have been calculated directly. We find

$$\text{(B9)} \qquad L_o = -\frac{1}{\pi\alpha}\left\{e^{\alpha}\mathrm{Ei}(-4\alpha) - e^{-\alpha}\mathrm{Ei}(4\alpha) - 2\alpha(\gamma + \ln 4\alpha)\sinh\alpha + 2(\sinh 2\alpha + 2\alpha)\sinh\alpha\right\}$$

and

$$\text{(B10)} \qquad L_1 = -\frac{1}{6\pi\alpha}\Big[\alpha^2(\alpha-2)e^{\alpha}\mathrm{Ei}(-4\alpha)-\alpha^2(\alpha+2)3^{-\alpha}\mathrm{Ei}(4\alpha)-2\alpha^2(\alpha\sinh\alpha - 2\cosh\alpha)$$

$$\cdot(\gamma + \ln 4\alpha) + 2(\alpha^2+2)\sinh\alpha\sinh 2\alpha$$

$$+ 4\alpha\left\{(\alpha^2 + 2)\sinh\alpha - 4\alpha\cosh\alpha\right\}\Big] .$$

Also

$$\text{(B11)} \qquad M_o = \frac{i}{\pi}\Big[4\cosh\alpha - \frac{2}{\alpha}\sinh\alpha - \frac{1}{\alpha^2}\sinh 2\alpha\sinh\alpha + e^{\alpha}\mathrm{Ei}(-4\alpha) - e^{-\alpha}\mathrm{Ei}(4\alpha)$$

$$- 2\cosh\alpha\,(\gamma + \ln 4\alpha)\Big]$$

and

$$\text{(B12)} \qquad M_1 = \frac{i}{6\pi}\Big[2\Big(\frac{2}{\alpha} + \alpha\Big)\sinh\alpha\cosh 2\alpha - \Big(5 + \frac{2}{\alpha^2}\Big)\sinh 2\alpha\sinh\alpha + 2\Big(\frac{1}{\alpha} - \alpha\Big)\sinh 2\alpha\cosh\alpha$$

$$- 16\alpha\sinh\alpha + 4(\alpha^2-1)\cosh\alpha - (2 + 2\alpha - \alpha^2)e^{\alpha}\mathrm{Ei}(-4\alpha)$$

$$+ (\alpha^2+2\alpha-2)e^{-\alpha}\mathrm{Ei}(4\alpha) + \left\{(2-\alpha^2)\cosh\alpha + 2\alpha\sinh\alpha\right\}(\gamma + \ln 4\alpha)\Big] .$$

It is helpful, when considering the behavior for small α, to have expansions in powers of α available. The first terms of the expansions of the above constants are therefore given below:

$$\text{(B13)} \qquad A_{oo} \approx \frac{2}{9}\alpha^3 + \frac{1}{75}\alpha^5 ,$$

$$\text{(B14)} \qquad A_{01} \approx \frac{\alpha^5}{45} ,$$

$$\text{(B15)} \qquad A_{11} \approx \frac{\alpha^7}{450} ,$$

$$K_o \approx \frac{4\alpha^4}{9} + \frac{26\alpha^6}{225} , \tag{B16}$$

$$K_1 \approx \frac{2\alpha^6}{45} , \tag{B17}$$

$$L_o \approx - \frac{\alpha^3}{\pi} \left(\frac{8}{9} + \frac{92\alpha^2}{225} \right), \tag{B18}$$

$$L_1 \approx - \frac{4\alpha^5}{45\pi} , \tag{B19}$$

$$M_o \approx \frac{4i\alpha^4}{15\pi} \left(2 + \frac{61\alpha^2}{105}\right) , \tag{B20}$$

and

$$M_1 \approx \frac{4i\alpha^6}{15\pi} \left(\frac{1}{5} + \frac{53\alpha^2}{882} \right) . \tag{B21}$$

Appendix C

In this appendix we prove the formula

$$\int_0^{\pi/2} J_{\nu-\frac{1}{2}}(z\sin\theta)\, J_{\mu+\frac{1}{2}}(Z\sin\theta)\, \sin^{\nu-\mu}\theta\, \cos^{2\lambda}\theta\, d\theta$$

(C1)

$$= z^{\nu-\frac{1}{2}} Z^{\lambda+\frac{1}{2}} \int_0^{\pi/2} J_{\mu-\lambda}(Z\sin\theta)\, \frac{J_{\nu+\lambda}\left\{(z^2-Z^2\cos^2\theta)^{1/2}\right\}}{(z^2-Z^2\cos^2\theta)^{(\nu+\lambda)/2}}\, \sin^{\mu-\lambda+1}\theta \cos^{2\lambda}\theta d\theta$$

when $\mathcal{R}(\lambda) > -\frac{1}{2}$, $\mathcal{R}(\mu) > \mathcal{R}(\lambda) - 1$, $\mathcal{R}(\nu) > -\frac{1}{2}$.

Sonine's second finite integral (Watson[13]) states that

$$\int_0^{\pi/2} J_\alpha(z\sin\theta)\, I_\beta(Z\cos\theta)\, \sin^{\alpha+1}\theta\, \cos^{\beta+1}\theta\, d\theta = z^\alpha Z^\beta\, \frac{J_{\alpha+\beta+1}\left\{(z^2-Z^2)^{1/2}\right\}}{(z^2-Z^2)^{(\alpha+\beta+1)/2}}$$

when $\mathcal{R}(\alpha) > -1$ and $\mathcal{R}(\beta) > -1$. Hence the left-hand side of (C1) can be written

$$Z\int_0^{\pi/2} \sin^{\nu+\frac{1}{2}}\theta\, \cos^{\lambda+\frac{1}{2}}\theta\, J_{\nu-\frac{1}{2}}(z\sin\theta) \int_0^{\pi/2} J_{\mu-\lambda}(Z\sin\phi) I_{\lambda-\frac{1}{2}}(Z\cos\theta\cos\phi)$$

$$\cdot \sin^{\mu-\lambda+1}\phi\, \cos^{\lambda+\frac{1}{2}}\phi\, d\phi\, d\theta$$

$$= Z\int_0^{\pi/2} \sin^{\mu-\lambda+1}\phi\, \cos^{\lambda+\frac{1}{2}}\phi\, J_{\mu-\lambda}(Z\sin\phi) \int_0^{\pi/2} J_{\nu-\frac{1}{2}}(z\sin\theta) I_{\lambda-\frac{1}{2}}(Z\cos\theta\cos\phi)$$

$$\cdot \sin^{\nu+\frac{1}{2}}\theta\, \cos^{\lambda+\frac{1}{2}}\theta\, d\theta d\phi$$

on inverting the order of integration. A further use of Sonine's second integral formula gives

$$z^{\nu-\frac{1}{2}} Z^{\lambda+\frac{1}{2}} \int_0^{\pi/2} \sin^{\mu-\lambda+1}\phi\, \cos^{2\lambda}\phi J_{u-\lambda}(Z\sin\phi)\, \frac{J_{\nu+\lambda}\left\{(z^2-Z^2\cos^2\theta)^{1/2}\right\}}{(z^2-Z^2\cos^2\theta)^{(\nu+\lambda)/2}}\, d\theta,$$

and (C1) is proved.

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